CONTINUOUS DISTRIBUTIONS

Laplace transform (Laplace-Stieltjes transform)

Definition

The Laplace transform of a non-negative random variable $X \geq 0$ with the probability density function $f(x)$ is defined as

$$\boxed{f^*(s) = \int_0^\infty e^{-st} f(t)dt = \mathrm{E}[e^{-sX}]} \quad = \int_0^\infty e^{-st} dF(t) \qquad \text{also denoted as } \mathcal{L}_X(s)$$

- Mathematically it is the Laplace transform of the pdf function.
- In dealing with continuous random variables the Laplace transform has the same role as the generating function has in the case of discrete random variables.
  - if X is a discrete integer-valued (≥ 0) r.v., then $f^*(s) = \mathcal{G}(e^{-s})$

Laplace transform of a sum

Let X and Y be independent random variables with L-transforms $f_X^*(s)$ and $f_Y^*(s)$.

$$\begin{aligned} f_{X+Y}^*(s) &= \mathrm{E}[e^{-s(X+Y)}] \\ &= \mathrm{E}[e^{-sX}e^{-sY}] \\ &= \mathrm{E}[e^{-sX}]\mathrm{E}[e^{-sY}] \qquad \text{(independence)} \\ &= f_X^*(s)f_Y^*(s) \end{aligned}$$

$$\boxed{f_{X+Y}^*(s) = f_X^*(s)f_Y^*(s)}$$

Calculating moments with the aid of Laplace transform

By derivation one sees

$$f^{*\prime}(s) = \frac{d}{ds}\mathrm{E}[e^{-sX}] = \mathrm{E}[-Xe^{-sX}]$$

Similarly, the n^{th} derivative is

$$f^{*(n)}(s) = \frac{d^n}{ds^n}\mathrm{E}[e^{-sX}] = \mathrm{E}[(-X)^n e^{-sX}]$$

Evaluating these at $s = 0$ one gets

$$\boxed{\begin{aligned} \mathrm{E}[X] &= -f^{*\prime}(0) \\ \mathrm{E}[X^2] &= +f^{*\prime\prime}(0) \\ &\vdots \\ \mathrm{E}[X^n] &= (-1)^n f^{*(n)}(0) \end{aligned}}$$

Laplace transform of a random sum

Consider the random sum

$$Y = X_1 + \cdots + X_N$$

where the X_i are *i.i.d.* with the common L-transform $f_X^*(s)$ and
$N \geq 0$ is a integer-valued r.v. with the generating function $\mathcal{G}_N(z)$.

$$\begin{aligned}
f_Y^*(s) &= \mathrm{E}[e^{-sY}] \\
&= \mathrm{E}[\mathrm{E}\left[e^{-sY} \,|\, N\right]] && \text{(outer expectation with respect to variations of } N\text{)} \\
&= \mathrm{E}[\mathrm{E}\left[e^{-s(X_1+\cdots+X_N)} \,|\, N\right]] && \text{(in the inner expectation } N \text{ is fixed)} \\
&= \mathrm{E}[\mathrm{E}[e^{-s(X_1)}]\cdots\mathrm{E}[e^{-s(X_N)}]] && \text{(independence)} \\
&= \mathrm{E}[(f_X^*(s))^N] \\
&= \mathcal{G}_N(f_X^*(s)) && \text{(by the definition } \mathrm{E}[z^N] = \mathcal{G}_N(z)\text{)}
\end{aligned}$$

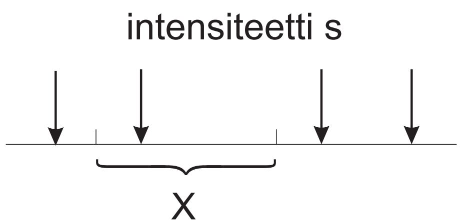
Laplace transform and the method of collective marks

We give for the Laplace transform

$$f^*(s) = \mathrm{E}[e^{-sX}], \quad X \geq 0, \qquad \text{the following}$$

Interpretation: Think of X as representing the length of an interval. Let this interval be subject to a Poissonian marking process with intensity s. Then the Laplace transform $f^*(s)$ is the probability that there are no marks in the interval.

$$\begin{aligned} \mathrm{P}\{X \text{ has no marks}\} &= \mathrm{E}[\mathrm{P}\{X \text{ has no marks} \,|\, X\}] && \text{(total probability)} \\ &= \mathrm{E}[\mathrm{P}\{\text{the number of events in the interval } X \text{ is } 0 \,|\, X\}] \\ &= \mathrm{E}[e^{-sX}] \;=\; f^*(s) \end{aligned}$$



$$\mathrm{P}\{\text{there are } n \text{ events in the interval } X \,|\, X\} = \frac{(sX)^n}{n!} e^{-sX}$$

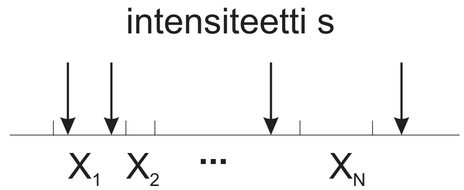
$$\mathrm{P}\{\text{the number of events in the interval } X \text{ is } 0 \,|\, X\} = e^{-sX}$$

Method of collective marks (continued)

Example: Laplace transform of a random sum

$$Y = X_1 + \cdots + X_N \ , \quad \text{where}$$

$$\begin{cases} X_1 \sim X_2 \sim \cdots \sim X_N, \ \text{common L-transform } f^*(s) \\ N \text{ is a r.v. with generating function } \mathcal{G}_N(z) \end{cases}$$



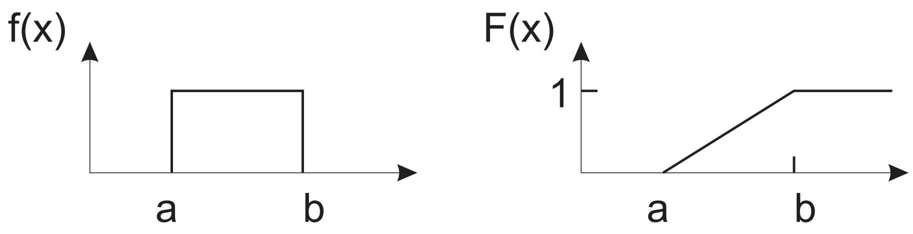
$$\begin{aligned} f_Y^*(s) &= \text{P}\{\text{none of the subintervals of } Y \text{ is marked}\} \\ &= \underbrace{\mathcal{G}_N(\underbrace{f_X^*(s)}_{\substack{\text{probability that a} \\ \text{single subinterval} \\ \text{has no marks}}})}_{\substack{\text{probability that none of} \\ \text{the subintervals is marked}}} \end{aligned}$$

Uniform distribution $X \sim \mathrm{U}(a, b)$

The pdf of X is constant in the interval (a, b):

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{elsewhere} \end{cases}$$

i.e. the value X is drawn randomly in the interval (a, b).



$$\mathrm{E}[X] = \int_{-\infty}^{+\infty} x f(x) dx = \frac{a+b}{2}$$

$$\mathrm{V}[X] = \int_{-\infty}^{+\infty} \left(x - \frac{a+b}{2}\right)^2 f(x) dx = \frac{(b-a)^2}{12}$$

Uniform distribution (continued)

Let $U_1, \ldots, U_n$ be independent uniformly distributed random variables, $U_i \sim \mathrm{U}(0,1)$.

- The number of variables which are $\leq x$ $(0 \leq x \leq 1))$ is $\sim \mathrm{Bin}(n,x)$
  - the event $\{U_i \leq x\}$ defines a Bernoulli trial where the probability of success is x
- Let $U_{(1)}, \ldots, U_{(n)}$ be the ordered sequence of the values.
  Define further $U_{(0)} = 0$ and $U_{(n+1)} = 1$.
  It can be shown that all the intervals are identically distributed and

  $$\mathrm{P}\{U_{(i+1)} - U_{(i)} > x\} = (1-x)^n \qquad i = 1, \ldots, n$$

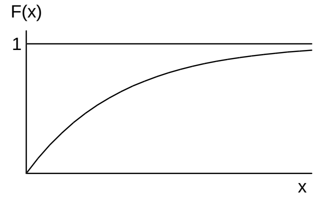
  - for the first interval $U_{(1)} - U_{(0)} = U_{(1)}$ the result is obvious because $U_{(1)} = \min(U_1, \ldots, U_n)$

Exponential distribution $X \sim \text{Exp}(\lambda)$

(Note that sometimes the shown parameter is $1/\lambda$, i.e. the mean of the distribution)

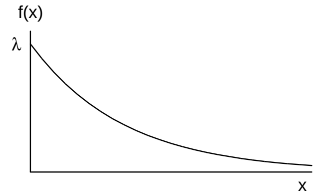
X is a non-negative continuous random variable with the cdf

$$F(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$



and pdf

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$



Example: interarrival time of calls; holding time of call

Laplace transform and moments of exponential distribution

The Laplace transform of a random variable with the distribution Exp(λ) is

$$f^*(s) = \int_0^\infty e^{-st} \cdot \lambda e^{-\lambda t} dt = \frac{\lambda}{\lambda + s}$$

With the aid of this one can calculate the moments:

$$\begin{aligned}
\mathrm{E}[X] &= -f^{*\prime}(0) &= \left.\tfrac{\lambda}{(\lambda+s)^2}\right|_{s=0} = \tfrac{1}{\lambda} \\
\mathrm{E}[X^2] &= +f^{*\prime\prime}(0) &= \left.\tfrac{2\lambda}{(\lambda+s)^3}\right|_{s=0} = \tfrac{2}{\lambda^2} \\
\mathrm{V}[X] &= \mathrm{E}[X^2] - \mathrm{E}[X]^2 &= \tfrac{1}{\lambda^2}
\end{aligned}$$

$$\boxed{\mathrm{E}[X] = \frac{1}{\lambda} \qquad \mathrm{V}[X] = \frac{1}{\lambda^2}}$$

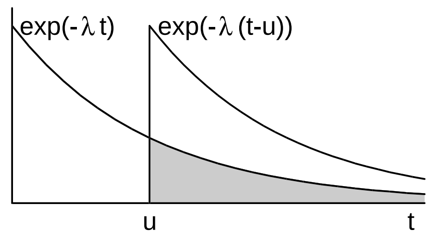
The memoryless property of exponential distribution

Assume that $X \sim \text{Exp}(\lambda)$ represents e.g. the duration of a call.

What is the probability that the call will last at least time x more given that it has already lasted the time t:

$$\begin{aligned} \text{P}\{X > t + x \,|\, X > t\} &= \frac{\text{P}\{X > t + x, X > t\}}{\text{P}\{X > t\}} \\ &= \frac{\text{P}\{X > t + x\}}{\text{P}\{X > t\}} \\ &= \frac{e^{-\lambda(t+x)}}{e^{-\lambda t}} = e^{-\lambda x} = \text{P}\{X > x\} \end{aligned}$$

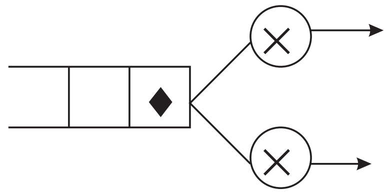
$$\boxed{\text{P}\{X > t + x \,|\, X > t\} = \text{P}\{X > x\}}$$



- The distribution of the remaining duration of the call does not at all depend on the time the call has already lasted
- Has the same $\text{Exp}(\lambda)$ distribution as the total duration of the call.

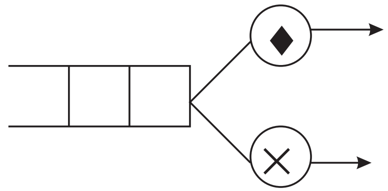
Example of the use of the memoryless property

A queueing system has two servers. The service times are assumed to be exponentially distributed (with the same parameter). Upon arrival of a customer ($\diamond$) both servers are occupied ($\times$) but there are no other waitng customers.



The question: what is the probability that the customer ($\diamond$) will be the last to depart from the system?

The next event in the system is that eithe of the customers ($\times$) being served departs and the customer enters ($\diamond$) the freed server.



By the memoryless property, from that point on the (remaining) service times of both customers ($\diamond$) and ($\times$) are identically (exponentially) distributed.

The situation is completely symmetric and consequently the probability that the customer ($\diamond$) is the last one to depart is 1/2.

The ending probability of an exponentially distributed interval

Assume that a call with Exp(λ) distributed duration has lasted the time t.
What is the probability that it will end in an infinitesimal interval of length h?

$$\begin{aligned} \mathrm{P}\{X \le t+h \,|\, X > t\} &= \mathrm{P}\{X \le h\} \quad \text{(memoryless)} \\ &= 1 - e^{-\lambda h} \\ &= 1 - (1 - \lambda h + \tfrac{1}{2}(\lambda h)^2 - \cdots) \\ &= \lambda h + o(h) \end{aligned}$$

The ending probability per time unit = λ

(constant!)

The minimum and maximum of exponentially distributed random variables

Let $X_1 \sim \cdots \sim X_n \sim \text{Exp}(\lambda)$ $\quad$ (*i.i.d.*)

The tail distribution of the minimum is

$$\begin{aligned} \text{P}\{\min(X_1,\ldots,X_n) > x\} &= \text{P}\{X_1 > x\}\cdots\text{P}\{X_n > x\} \qquad \text{(independence)} \\ &= (e^{-\lambda x})^n = e^{-n\lambda x} \end{aligned}$$

The minimum obeys the distribution $\text{Exp}(n\lambda)$.

The ending intensity of the minimum $= n\lambda$

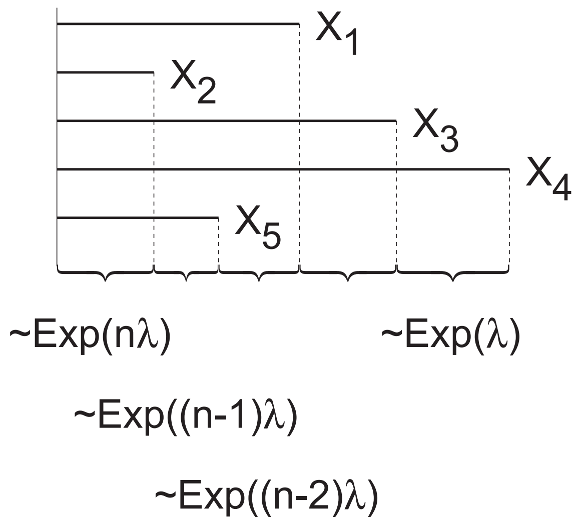
n parallel processes each of which ends with intensity λ independent of the others

The cdf of the maximum is

$$\text{P}\{\max(X_1,\ldots,X_n) \le x\} = (1 - e^{-\lambda x})^n$$

The expectation can be deduced by inspecting the figure

$$\text{E}[\max(X_1,\ldots,X_n)] = \frac{1}{n\lambda} + \frac{1}{(n-1)\lambda} + \cdots + \frac{1}{\lambda}$$



Erlang distribution $X \sim \text{Erlang}(n, \lambda)$

Also denoted Erlang-$n(\lambda)$.

X is the sum of n independent random variables with the distribution $\text{Exp}(\lambda)$

$$X = X_1 + \cdots + X_n \qquad X_i \sim \text{Exp}(\lambda) \quad (i.i.d.)$$

The Laplace transform is

$$f^*(s) = (\frac{\lambda}{\lambda + s})^n$$

By inverse transform (or by recursively convoluting the density function) one obtains the pdf of the sum X

$$\boxed{f(x) = \frac{(\lambda x)^{n-1}}{(n-1)!} \lambda e^{-\lambda x}} \qquad x \geq 0$$

Erlang distribution (continued): gamma distribution

The formula for the pdf of the Erlang distribution can be generalized, from the integer parameter n, to arbitrary real numbers by replacing the factorial $(n-1)!$ by the gamma function $\Gamma(n)$:

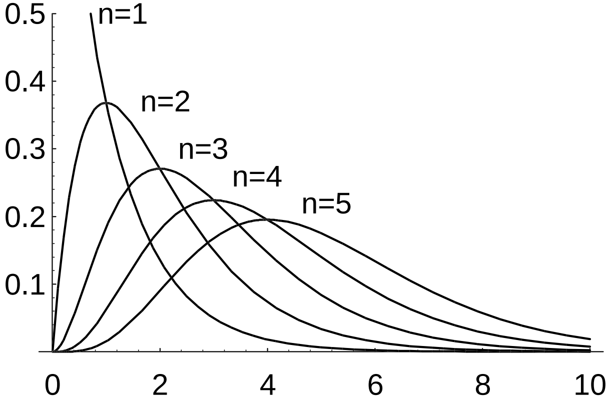
$$f(x) = \frac{(\lambda x)^{p-1}}{\Gamma(p)} \lambda e^{-\lambda x}$$

Gamma(p, λ) distribution

Gamma function $\Gamma(p)$ is defined by

$$\Gamma(p) = \int_0^\infty e^{-u} u^{p-1} du$$

By partial integration it is easy to see that when p is an integer then, indeed, $\Gamma(p) = (p-1)!$



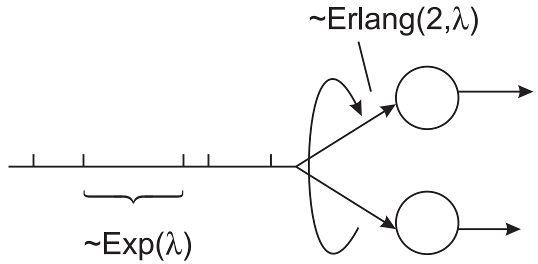
The expectation and variance are n times those of the Exp(λ) distribution:

$$\mathrm{E}[X] = \frac{n}{\lambda} \qquad \mathrm{V}[X] = \frac{n}{\lambda^2}$$

Erlang distribution (continued)

Example. The system consists of two servers. Customers arrive with Exp(λ) distributed interarrival times. Customers are alternately sent to servers 1 and 2.

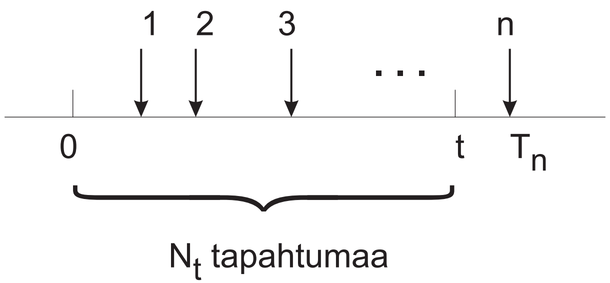
The interarrival time distribution of customers arriving at a given server is Erlang($2, \lambda$).



Proposition. Let N_t, the number of events in an interval of length t, obey the Poisson distribution:

$$N_t \sim \text{Poisson}(\lambda t)$$

Then the time T_n from an arbitrary event to the n^{th} event thereafter obeys the distribution Erlang(n, λ).



Proof.

$$\begin{aligned} F_{T_n}(t) &= \mathrm{P}\{T_n \le t\} = \mathrm{P}\{N_t \ge n\} \\ &= \sum_{i=n}^{\infty} \mathrm{P}\{N_t = i\} = \sum_{i=n}^{\infty} \frac{(\lambda t)^i}{i!} e^{-\lambda t} \end{aligned}$$

$$\begin{aligned} f_{T_n} = \frac{d}{dt} F_{T_n}(t) &= \sum_{i=n}^{\infty} \frac{i\lambda\,(\lambda t)^{i-1}}{i!} e^{-\lambda t} - \sum_{i=n}^{\infty} \frac{(\lambda t)^i}{i!} \lambda e^{-\lambda t} \\ &= \sum_{i=n}^{\infty} \frac{(\lambda t)^{i-1}}{(i-1)!} \lambda e^{-\lambda t} - \sum_{i=n}^{\infty} \frac{(\lambda t)^i}{i!} \lambda e^{-\lambda t} \\ &= \frac{(\lambda t)^{n-1}}{(n-1)!} \lambda e^{-\lambda t} \end{aligned}$$

Normal distribution $X \sim \mathrm{N}(\mu, \sigma^2)$

The pdf of a normally distributed random variable X with parameters μ ja σ^2 is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma}\, e^{-\frac{1}{2}(x-\mu)^2/\sigma^2}$$

Parameters μ and σ^2 are the expectation and variance of the distribution

$$\begin{cases} \mathrm{E}[X] = \mu \\ \mathrm{V}[X] = \sigma^2 \end{cases}$$

Proposition: If $X \sim \mathrm{N}(\mu, \sigma^2)$, then $Y = \alpha X + \beta \sim \mathrm{N}(\alpha\mu + \beta, \alpha^2\sigma^2)$.

Proof:

$$\begin{aligned} F_Y(y) &= \mathrm{P}\{Y \le y\} = \mathrm{P}\{X \le \tfrac{y-\beta}{\alpha}\} = F_X(\tfrac{y-\beta}{\alpha}) \\ &= \int_{-\infty}^{(y-\beta)/\alpha} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(x-\mu)^2/\sigma^2} dx \qquad z = \alpha x + \beta \\ &= \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}(\alpha\sigma)} e^{-\frac{1}{2}(z-(\alpha\mu+\beta))^2/(\alpha\sigma)^2} dz \end{aligned}$$

Seuraus: $Z = \dfrac{X-\mu}{\sigma} \sim \mathrm{N}(0,1) \qquad (\alpha = 1/\sigma,\ \beta = -\mu/\sigma)$

Denote the pdf of a N(0,1) random variable by $\Phi(x)$. Then

$$F_X(x) = \mathrm{P}\{X \le x\} = \mathrm{P}\{Z \le \tfrac{x-\mu}{\sigma}\} = \Phi(\tfrac{x-\mu}{\sigma})$$

Multivariate Gaussian (normal) distribution

Let $X_1, \ldots, X_n$ be a set of Gaussian (i.e. normally distributed) random variables with expectations $\mu_1, \ldots, \mu_n$ and covariance matrix

$$\mathbf{\Gamma} = \begin{pmatrix} \sigma_{11}^2 & \cdots & \sigma_{1n}^2 \\ \vdots & \ddots & \vdots \\ \sigma_{n1}^2 & \cdots & \sigma_{nn}^2 \end{pmatrix} \qquad \sigma_{ij}^2 = \mathrm{Cov}[X_i, X_j] \qquad (\sigma_{ii}^2 = \mathrm{V}[X_i])$$

Denote $\mathbf{X} = (X_1, \ldots, X_n)^{\mathrm{T}}$.

The probability density function of the random vector $\mathbf{X}$ is

$$\boxed{f(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n |\mathbf{\Gamma}|}} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}}\mathbf{\Gamma}^{-1}(\mathbf{x}-\boldsymbol{\mu})}}$$

where $|\mathbf{\Gamma}|$ is the determinant of the covariance matrix.

By a change of variables one sees easily that the pdf of the random vector $\mathbf{Z} = \mathbf{\Gamma}^{-1/2}(\mathbf{X} - \boldsymbol{\mu})$ is $(2\pi)^{-n/2}\exp(-\frac{1}{2}\mathbf{z}^{\mathrm{T}}\mathbf{z}) = \sqrt{2\pi}e^{-z_1^2/2} \cdots \sqrt{2\pi}e^{-z_n^2/2}$.

Thus the components of the vector $\mathbf{Z}$ are independent N(0,1) distributed random variables.

Conversely, $\boxed{\mathbf{X} = \boldsymbol{\mu} + \mathbf{\Gamma}^{1/2}\mathbf{Z}}$ by means of which one can generate values for $\mathbf{X}$ in simulations.