Loss system

Consider a loss system, where the following parameters are given

$$\begin{cases} n = \text{number of trunks (elements which are reserved)} \\ a = \text{the intensity of the offered traffic} \end{cases}$$

There are no waiting places in the system. Calls which upon arrival find all trunks reserved are blocked and lost.

Question: what is the probability with which an arriving call is blocked?

- Time blocking refers to the proportion of time the system spends in the blocking state where all n elements are reserved.
- Call blocking is the proportion of the arriving calls which are blocked.
- Traffic blocking is the ratio of the traffic intensity of the blocked traffic to that of the offered traffic.

All these quantities are equal if

- the arrival process is Poisson
- the service (holding) times of the calls are independent and identically distributed

Blocking in the $M/M/n/n$ system: Erlang's formula

Assume that the arrival process of the customers arrive according to a Poisson process with intensity λ and that the service time obeys the distribution $\mathrm{Exp}(\mu)$

$$\begin{cases} \lambda &= \text{the arrival intensity (rate) of the customers} \\ \mu &= \text{the service rate of the server (the mean service time is } 1/\mu) \end{cases}$$

Denote

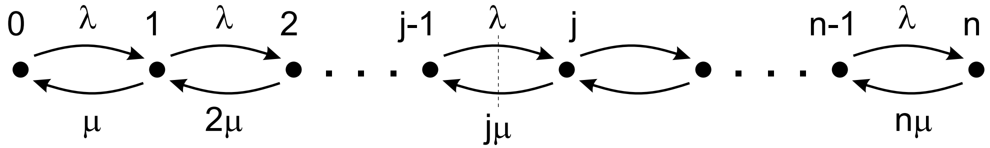
$$\begin{cases} N &= \text{number of elements reserved (number of customers in system)} \\ \pi_j &= \mathrm{P}\{N = j\} \text{ the equilibrium probability of state } j \end{cases}$$

The state variable N_t constitutes a Markov process of the birth-death type

- the state can change only stepwise

Erlang's formula (continued)

The state transition diagram is



The balance across the cut leads to the recursion

$$\lambda \pi_{j-1} = j\mu\pi_j \qquad \text{eli} \qquad \pi_j = \frac{a}{j}\,\pi_{j-1} \qquad (a = \lambda/\mu = \text{offered traffic intensity})$$

By repeated application of the recursion one obtains

$$\pi_j = \frac{a^j}{j!}\,\pi_0, \quad j = 0, 1, \ldots, n$$

From the normalization condition $\pi_0 + \pi_1 \ldots + \pi_n = 1$ one can solve $\pi_0 = 1/(1 + \frac{a}{1!} + \frac{a^2}{2!} + \ldots + \frac{a^n}{n!})$

$$\boxed{\pi_j = \frac{\dfrac{a^j}{j!}}{1 + \dfrac{a}{1!} + \dfrac{a^2}{2!} + \ldots + \dfrac{a^n}{n!}}}$$

- the equilibrium probabilities in the $M/M/n/n$ system
- truncated Poisson distribution
- $\to \dfrac{a^j}{j!}\,e^{-a}$, as $n \to \infty$

Erlang's formula (continued)

The prob. π_n of the state n gives the time blocking prob. (= call blocking prob. in this model). We get the celebrated Erlang's formula (also called Erlang's B-formula):

$$\boxed{E(n,a) = \frac{\dfrac{a^n}{n!}}{1 + \dfrac{a}{1!} + \dfrac{a^2}{2!} + \ldots + \dfrac{a^n}{n!}}}$$

A canonical traffic theoretical relation: it relates the size of the system n, the offered traffic a and the experienced quality of service (blocking).

Example.

A modem pool consists of 4 modems and the offered traffic intensity is 2 erl.
What is the probability that a connection attempt is fails due to blocking?
What is the blocking probability, if the number of modems is increased to 6?

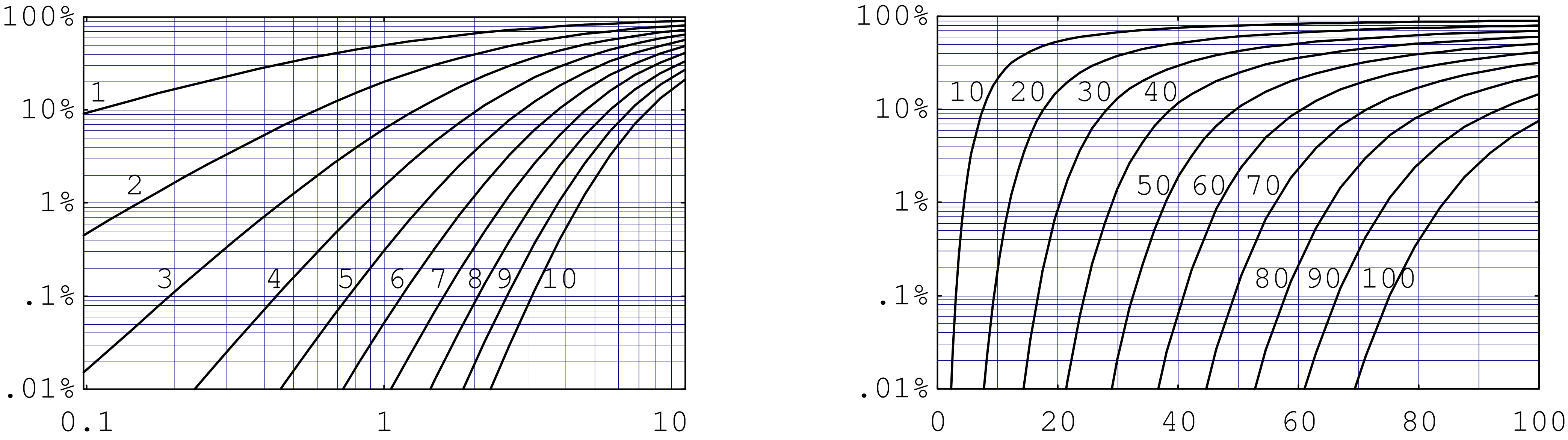
Answer: the original blocking probability is 9.5 % and after the increase of the number of modems it is 1.2 %.

Insensitivity

Erlang's formula holds more generally independent of the form of the service time distribution.

The blocking depends only on mean holding time $1/\mu$ through the traffic intensity $a = \lambda/\mu$.

Graphs for Erlang's blocking function



- Horizontal axis: the offered traffic intensity a
- The parameter of the family of curves: the size of the system n
- Vertical axis: blocking probability $E(n, a)$

The required capacity as a function of the load

The following table gives the required number of trunks n as a function of the offered traffic intensity a when the allowed blocking is 1 %. The last column gives the required relative oversizing n/a, i.e. the ratio of the number of trunks to the load.

a (erl)	n	n/a
3	8	2.7
10	18	1.8
30	42	1.4
100	117	1.17
300	324	1.08
1000	1029	1.03

- When the traffic intensity a is large the Poisson fluctuations in the occupancy are small in relative terms, and the required oversizing is small
  - for Poisson distribution the standard deviation to mean ratio is $\sqrt{a}/a = 1/\sqrt{a}$.
- From the point of view of dimensioning the system it is then (a large) more important that the value of a on which the dimensioning is based has been correctly estimated and the uncertainties in it have been properly accounted for.

Recursion formula

Erlang's function is given by a simple expression which is easy to evaluate.

Some planning tools, however, make very frequent calls to this function, possibly in nested iterative loops. Then it is important to pay attention to the fast computation of the function. Often the following recursion is advantageous:

$$E(n,a) = \frac{\frac{a^n}{n!}}{1+\frac{a}{1!}+\frac{a^2}{2!}+\cdots+\frac{a^n}{n!}}$$

$$\frac{E(n,a)}{E(n-1,a)} = \frac{\frac{a^n}{n!}}{\frac{a^{n-1}}{(n-1)!}} \frac{1+\frac{a}{1!}+\frac{a^2}{2!}+\cdots+\frac{a^{n-1}}{(n-1)!}}{1+\frac{a}{1!}+\frac{a^2}{2!}+\cdots+\frac{a^n}{n!}} = \frac{a}{n}\Big(1 - \underbrace{\frac{\frac{a^n}{n!}}{1+\frac{a}{1!}+\frac{a^2}{2!}+\cdots+\frac{a^n}{n!}}}_{E(n,a)}\Big)$$

$$E(n,a) = \frac{a}{n}E(n-1,a)(1-E(n,a)) \quad \Rightarrow \quad E(n,a)(n+aE(n-1,a)) = aE(n-1,a)$$

$$\boxed{\begin{aligned} E(0,a) &= 1 \\ E(n,a) &= \frac{aE(n-1,a)}{n+aE(n-1,a)} \end{aligned}} \qquad \boxed{\begin{aligned} F(0,a) &= 1 \\ F(n,a) &= 1+\frac{n}{a}F(n-1,a) \end{aligned}}$$

The latter form has been obtained by writing the recursion for the inverse $F(n,a) = 1/E(n,a)$. In this recursion, one first computes $F(n,a)$ from which one obtains $E(n,a) = 1/F(n,a)$.

The insensitivity of the equilibrium distribution

Above we have derived the result that the equilibrium probability distribution is a truncated Poisson distribution:

$$\boxed{\pi_j = \frac{\dfrac{a^j}{j!}}{1 + \dfrac{a}{1!} + \dfrac{a^2}{2!} + \ldots + \dfrac{a^n}{n!}}} \qquad j = 0, 1, \ldots, n$$

$$\begin{cases} a & = \lambda \bar{X} \\ \lambda & = \text{Poisson arrival intensity} \\ \bar{X} & = \text{mean holding time } (1/\mu) \end{cases}$$

The derivation was based on the assumption that the arrival process is Poissonian and that the holding time obeys exponential distribution.

Remarkably, however, the result holds more generally: the <u>insensitivity</u> result.

The formula for the equilibrium probabilities (and in particular for the blocking probability π_n) is valid for any holding time distribution and depends on the distribution through the mean holding time $\bar{X}$ only. (Poisson assumption, however, is necessary.)

- The proof of the insensitivity is a non-trivial task in the general case.
- In the following we make considerations by which verify the insensitivity in the cases $n = 1$ and $n = \infty$.

Insensitivity in the case $n = 1$ (the $M/M/1/1$ system)

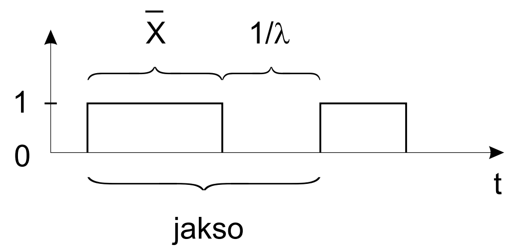
In the case $n = 1$ the truncated Poisson distribution reduces to the form

$$\boxed{\pi_0 = \frac{1}{1+a}, \qquad \pi_1 = \frac{a}{1+a}, \qquad \text{where } a = \lambda \bar{X}}$$

The insensitivity claim: The state probabilities are valid irresespective of the form of the holding time distribution.

Proof: The state of the system alternates between "server busy" and "server idle".

Consider a full cycle which consists of one reserved period and one idle period.



All cycles are stochastically identical. The probability of the busy/idle state equals the average proportion of the busy/idle period of the total length the total period.

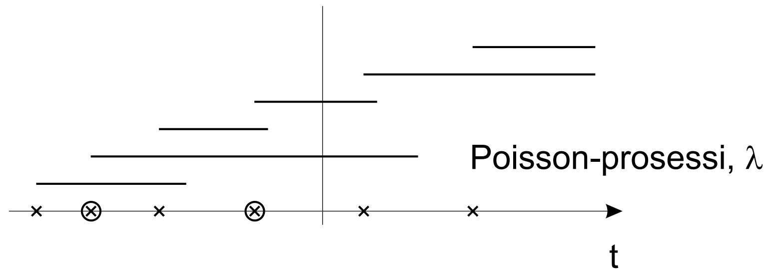
$$\begin{cases} \bar{X} & = \text{expected duration of a busy period} \\ 1/\lambda & = \text{expected duration of an idle period} \\ & \text{(the interarrival times are distributed according to Exp}(\lambda)\text{; memoryless!)} \\ \bar{X} + 1/\lambda & = \text{expected duration of the period} \end{cases}$$

$$\pi_0 = \frac{1/\lambda}{\bar{X} + 1/\lambda} = \frac{1}{1+a}, \quad \pi_1 = \frac{\bar{X}}{\bar{X} + 1/\lambda} = \frac{a}{1+a}$$

Knowledge about the distribution of X was not needed!

Insensitivity in the case $n = \infty$ (the $M/M/\infty/\infty$ system)

Now the system is non-blocking. Consider the number of calls in progress at time 0. This is equal to the number of starting events (before 0) of calls which are still in progress at time 0.



- The call durations X for all calls are independent: tail distribution $G(t) = \mathrm{P}\{X > t\}$.
- Select all the calls that extend over 0. A call arriving at time $t < 0$ is selected with the probability $G(-t)$.
- The arrival process of the selected calls is an inhomogeneous Poisson process with intensity $\lambda(t) = \lambda \cdot G(-t)$.
- The number of calls in progress at time 0 equals the number of arrivals from the inhomogeneous Poisson process in the interval $(-\infty, 0)$. The number is $\sim$ Poisson(a), where $a = \int_{-\infty}^{0} \lambda(t)dt$.

$$a = \int_{-\infty}^{0} \lambda G(-t)dt = \lambda \int_0^\infty G(t)dt = \lambda\Big(\underbrace{\big/_0^\infty t\, G(t)}_{0} - \int_0^\infty \underbrace{G'(t)}_{-f(t)}\, t\, dt\Big)$$

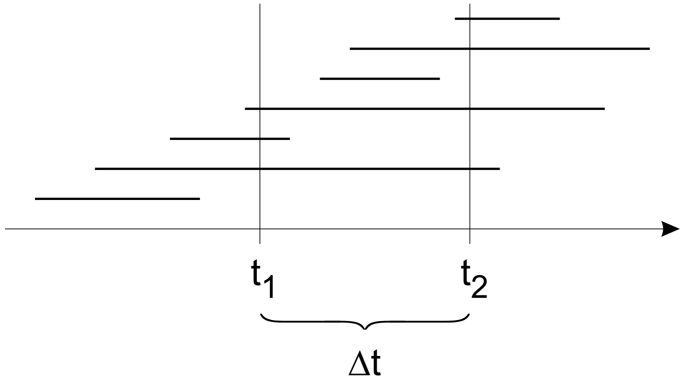
$$= \lambda \int_0^\infty t f(t)\, dt = \lambda \bar{X}$$

The number is distributed as $\boxed{\text{Poisson}(\lambda\bar{X})}$ which depends on the holding time only through the mean $\bar{X}$.

The covariance of the occupancy states at two different instants of time in the $M/M/\infty/\infty$ system

Consider the covariance between the values of the occupancy N at two different instants t_1 ja t_2.

- Calls (customers) arrive with a Poissonian intensity λ.
- The tail distribution of the holding time X is $G(x) = \mathrm{P}\{X > x\}$ (general distribution).



Denote

$$\begin{cases} N_1 = \text{number of calls at time } t_1 \\ N_2 = \text{number of calls at time } t_2 \end{cases}$$

We wish to calculate $\mathrm{Cov}[N_1, N_2]$.

- If t_1 and t_2 are two close instants, one can assume that $N_1 \approx N_2$ and $\mathrm{Cov}[N_1, N_2] \approx \mathrm{V}[N] = \lambda \bar{X} = a$.
- If t_1 and t_2 are far apart, N_1 and N_2 are approximately independent and $\mathrm{Cov}[N_1, N_2] \approx 0$.

Covariance of the occupations (continued)

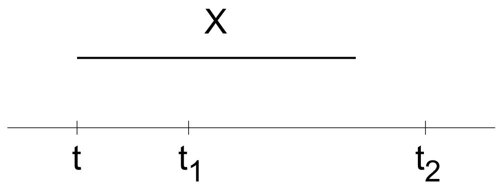
We split both N_1 and N_2 into independent components as follows

$$\begin{cases} N_1 &= K_1 + K_{1,2} \\ N_2 &= K_2 + K_{1,2} \end{cases}$$

whence the covariance arises only through the common component $K_{1,2}$,

$$\begin{aligned} \mathrm{Cov}[N_1, N_2] &= \mathrm{Cov}[K_1 + K_{1,2}, K_2 + K_{1,2}] \\ &= \mathrm{Cov}[K_1, K_2] + \mathrm{Cov}[K_1, K_{1,2}] + \mathrm{Cov}[K_{1,2}, K_2] + \mathrm{Cov}[K_{1,2}, K_{1,2}] \\ &= \mathrm{Cov}[K_{1,2}, K_{1,2}] = \mathrm{V}[K_{1,2}] \end{aligned}$$

The component K_1: the number of calls which are in progress at time t_1 but not at time t_2

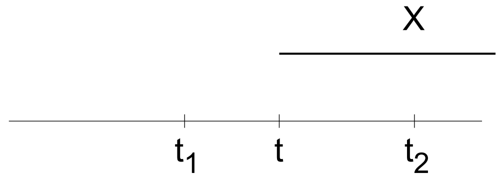


- an arrival at $t < t_1$
- condition for the duration X: $t_1 - t < X < t_2 - t$
- random selection from a Poisson stream λ in $-\infty < t < t_1$ with the probability $G(t_1 - t) - G(t_2 - t)$

Covariance of the occupations (continued)

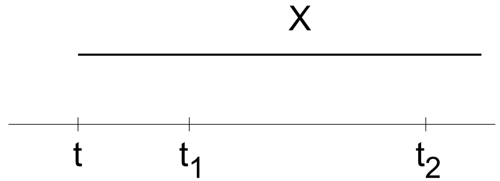
<ins>The component K_2</ins>: the number of calls which are in progress at t_2 but which start after t_1

- arrival at $t_1 < t < t_2$
- the condition for the duration X: $X > t_2 - t$
- random selection from a Poisson stream λ in $t_1 < t < t_2$ with the probability $G(t_2 - t)$



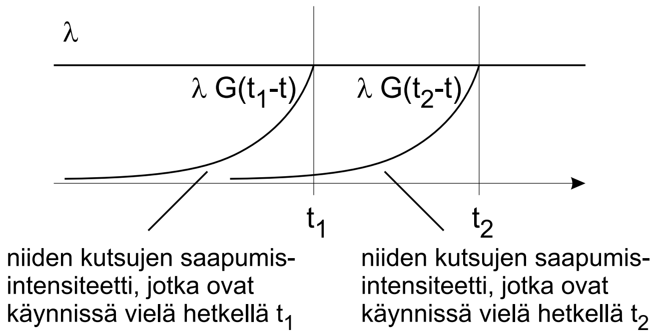
<ins>Component $K_{1,2}$</ins>: the number of calls which are in progress both at t_1 and at t_2

- arrival at $t < t_1$
- the condition for the duration X: $X > t_2 - t$
- random selection from a Poisson stream λ in $t < t_1$ with the probability $G(t_2 - t)$

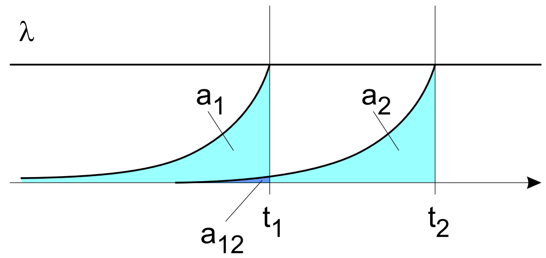


Covariance of the occupations (continued)

The quantities K_1, K_2 ja $K_{1,2}$ represent total number of arrivals from inhomogeneous Poisson processes, which result from random selection or random split in the respective intervals.



By the general results on random selection / random split, K_1, K_2 and $K_{1,2}$ all obey Poisson distribution and are independent,



$$\begin{cases} K_1 \sim \text{Poisson}(a_1), & a_1 = \lambda\int_{-\infty}^{t_1}(G(t_1-t)-G(t_2-t))dt \\ K_2 \sim \text{Poisson}(a_2), & a_2 = \lambda\int_{t_1}^{t_2} G(t_2-t)dt \\ K_{1,2} \sim \text{Poisson}(a_{1,2}), & a_{1,2} = \lambda\int_{-\infty}^{t_1} G(t_2-t)dt \end{cases}$$

Since $\text{V}[K_{1,2}] = a_{1,2}$ we obtain (change of variable $x = t_2 - t$)

$$\boxed{\text{Cov}[N_1, N_2] = \lambda\int_{t_2-t_1}^{\infty} G(t)dt}$$

When $t_1 = t_2$, we find that $\text{V}[N_1] = \text{Cov}[N_1, N_1] = \lambda\overbrace{\int_0^\infty G(t)dt}^{\overbrace{\big/_0^\infty t\,G(t)}^{0} - \int_0^\infty t\,\overbrace{G'(t)}^{-f(t)}\,dt} = \lambda\bar{X} = a$, as it should.