TIME REVERSAL AND REVERSIBLE PROCESSES

Reversed process

Consider a irreducible stationary stochastic process X_t.

To this process one can associate so called reversed process X_t^*, where the process X_t is considered in the reversed time (" the film is run backwards").

$$X_t^* = X_{\tau-t}$$

Time reversal (mirroring) with respect to time τ.
The parameter τ is unimportant; it only defines where in the reversed process the origin of time is located.

Why do we study the reversed process

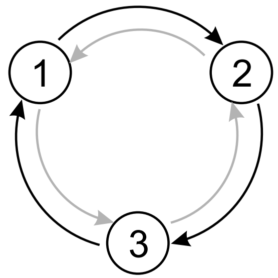
- It turns out that with the aid of it one can get more insight on the properties of the process.
- By considering the reversed process, one can often very simply and elegantly derive results, a direct derivation of which might be quite complicated.
- For instance, the balance equations of a complex system can be derived by "guessing" the reversed process.
- The departure process (output) of a queue can often most simply be analyzed by studying the reversed process.

Reversed process (continued)

In general, the reversed process X_t^* is a different process from the original one.

Example: Cyclic (periodical) process.

In the process X_t, the states appear in the order $1 \to 2 \to 3 \to 1$. In the reversed process, the sequence is $1 \to 3 \to 2 \to 1$. Obviously these must be two different processes.



The equilibrium distribution of the reversed process

Assume that the process X_t has the equilibrium distribution $\pi_i = \mathrm{P}\{X_t = i\}$.

Then also the reversed process X_t^* has an equilibrium distribution $\pi_i^* = \mathrm{P}\{X_t^* = i\}$ and this distribution is the same as for the original process

$$\boxed{\pi_i^* = \pi_i \quad \forall i}$$

Proof. π_i and π_i^* represent the proportions of time the processes X_t and X_t^* spend in state i. This proportion of time is independent of in which direction of time we consider the process.

Reversible process

If the reversed process X_t^* and the original process X_t are statistically indistinguishable, one says that the process X_t is time reversible.

More precisely, the reversibility means that

$$(X_{t_1}, X_{t_2}, \ldots, X_{t_n}) \sim (X_{\tau-t_1}, X_{\tau-t_2}, \ldots, X_{\tau-t_n}) \qquad \text{for all } t_1, t_2, \ldots, t_n \text{ and } \tau \text{ and } n$$

i.e. the shown sets of values of random variables have the same joint distributions.

- Intuitively, the reversibility of the process X_t means that an outside observer cannot tell whether the film is run in forward or backward direction.

Markov chain in reversed time

Proposition. The reversed chain $\ldots, X_{n+1}, X_n, X_{n-1}, \ldots$ of a Markov stationary chain $\ldots, X_{n-1}, X_n, X_{n+1}, \ldots$ constitutes a stationary Markov chain.

Proof. Consider the probability of the value $X_m = j$ conditioned on the following values $X_{m+1} = i, X_{m+2} = i_2, \ldots, X_{m+k}$ in the chain, which in the reversed time are previous values:

$$
\begin{aligned}
&\mathrm{P}\{X_m = j \,|\, X_{m+1} = i, X_{m+2} = i_2, \ldots, X_{m+k} = i_k\} \\
&= \frac{\mathrm{P}\{X_m = j, X_{m+1} = i, X_{m+2} = i_2, \ldots, X_{m+k} = i_k\}}{\mathrm{P}\{X_{m+1} = i, X_{m+2} = i_2, \ldots, X_{m+k} = i_k\}} \\
&= \frac{\mathrm{P}\{X_m = j, X_{m+1} = i\}\mathrm{P}\{X_{m+2} = i_2, \ldots, X_{m+k} = i_k \,|\, \overbrace{X_m = j}^{\text{doesn't depend on this, Markov!}}, X_{m+1} = i\}}{\mathrm{P}\{X_{m+1} = i\}\mathrm{P}\{X_{m+2} = i_2, \ldots, X_{m+k} = i_k \,|\, X_{m+1} = i\}} \\
&= \frac{\mathrm{P}\{X_m = j, X_{m+1} = i\}}{\mathrm{P}\{X_{m+1} = i\}} \\
&= \frac{\mathrm{P}\{X_m = j\}\mathrm{P}\{X_{m+1} = i \,|\, X_m = j\}}{\mathrm{P}\{X_{m+1} = i\}} = \frac{\pi_j p_{j,i}}{\pi_i}
\end{aligned}
$$

does not depend on the values of $X_{m+2} \ldots X_{m+k}$, $i_2, \ldots, i_k$

The transition probabilities of the reversed process are

$$
\boxed{p^*_{i,j} = \mathrm{P}\{X_m = j \,|\, X_{m+1} = i\} = \frac{\pi_j p_{j,i}}{\pi_i}}
$$

Markov process in reversed time (continued)

Proposition. Let X_t be a stationary (continuous time) Markov process with state transition rates $q_{i,j}$ and equilibrium probabilities π_i. Then the reversed process X_t^* is a stationary Markov process and its transition rates are

$$\boxed{q^*_{i,j} = \frac{\pi_j q_{j,i}}{\pi_i}}$$

Proof. Similar to that for the Markov chains.

Note. These propositions establish only that the reversed process is also Markovian, and not that it were identical to the original one. Reversibility in an additional property.

Solving equilibrium probabilities with the aid of the reversed process

Proposition. Let X_t be a Markov process with the transition rates $q_{i,j}$. If there are numbers $q^*_{i,j}$ and π_i such that

$$\sum_{j\neq i} q_{i,j} = \sum_{j\neq i} q^*_{i,j} \quad \forall i \qquad \text{and} \qquad \pi_i q^*_{i,j} = \pi_j q_{j,i} \quad \forall i,j \qquad \text{and} \qquad \sum_i \pi_i = 1$$

then the π_i are the common equilibrium probabilities of the processes X_t and X^*_t and the $q^*_{i,j}$ are the transition rates of the process X^*_t.

Proof:

$$\sum_{j\neq i} \pi_j q_{j,i} = \sum_{j\neq i} \pi_i q^*_{i,j} = \pi_i \sum_{j\neq i} q^*_{i,j} = \pi_i \sum_{j\neq i} q_{i,j} = \sum_{j\neq i} \pi_i q_{i,j} \qquad \forall i$$

Thus the π_i satisfy the global balance equations of the process X_t. Moreover, $q^*_{i,j} = \pi_j q_{i,j}/\pi_i$ i.e. the transition rate of the process X^*_t.

- This result can (somewhat surprisingly) been used to prove that a guessed distribution π_i indeed is the equilibrium distribution, by additionally guessing the transition rates $q^*_{i,j}$ of the reversed process.
- There are indeed problems (some complex systems), where the π_i can be guessed and the reversed process is rather obvious, and where the direct check of the balance condition would require a lot of work.

Reversible Markov process

The reversed Markov process X_t^* behaves as the original one if it has the same transition rates. Thus, the condition for the reversibility is

$$\boxed{q_{i,j}^* = q_{i,j}} \qquad \forall i, j$$

By the expression given earlier for $q_{i,j}^*$ the condition is equivalent to the following property

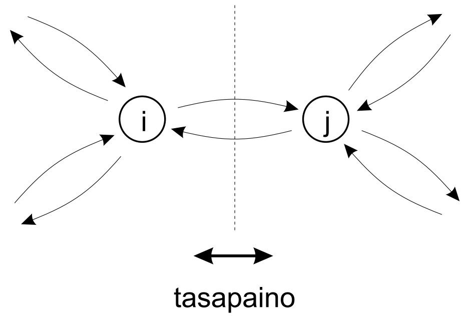
$$\boxed{\pi_i q_{i,j} = \pi_j q_{j,i}} \qquad \forall i, j$$

<u>Detailed balance</u>
condition for reversibility

- The detailed balance says that the probability flows between any two states are in balance.
- Detail balance implies immediately the global balance, i.e. the total probability flow out of a state, $\sum_j \pi_i q_{i,j}$, equals the total flow into the state, $\sum_j \pi_j q_{j,i}$.
- Further, it follows that if there exist numbers π_i such that the detailed balance conditions are satisfied, then the π_i are the equilibrium probabilities of the system (normalized $\sum_i \pi_i = 1$).
- The converse is not true, global balance does not imply detailed balance; all Markov processes are not reversible.

Detailed balance

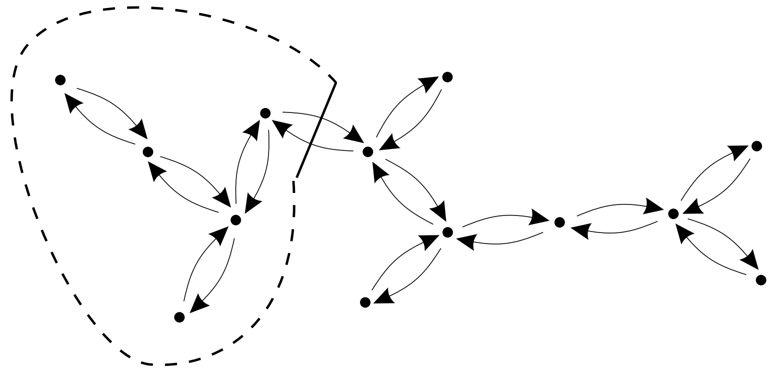
$$\begin{cases} \pi_i q_{i,j} = \text{frequence of transitions } i \to j \\ \pi_j q_{j,i} = \text{frequence of transitions } j \to i \end{cases}$$



The detailed balance says that in the processes X_t the transition frequencies between the states i and j are the same in both directions.

- In a reversible process this must be true, since by reversing the time the transition $i \to j$ becomes the transition $j \to i$ and vice versa. In order for the reversed process look the same as the original one, the transition frequencies in both directions must be equal.

Trees are reversible



Proposition: If the state transition diagram of a Markov process is a tree, then the process is time reversible.

Proof. By making a cut between any two states the tree is divided into two separate parts. From the global balance condition applied on these two sets of states it follows that the probability flows across the cut, i.e. between the two states, satisfy the detailed balance.

Corollary: All Markov processes of the birth-death type are time reversible.

Example. $M/M/1$, $M/M/n$, $M/M/\infty$, $M/M/m/m$, ...

Kolmogorov criterion

The reversibility condition expressed in the form of the detailed balance condition can be applied only when both the transition rates $q_{i,j}$ and the equilibrium probabilities π_i are known.

The equilibrium probabilities can always be solved when the $q_{i,j}$ are given, and thus in order to check the validity of the detailed balance it is sufficient to know the transition rates $q_{i,j}$.

One may wonder whether the reversibility (detailed balance) can be inferred more directly from the transition rates $q_{i,j}$ without first solving the equilibrium probabilities. The answer is yes, and is more specifically given by:

Kolmogorov criterion

Let $i_1, i_2, \ldots, i_m, i_1$ be a closed cycle in the transition diagram. The Kolmogorov criterion is satisfied if for every such cycle the following holds

$$q_{i_1,i_2} \cdot q_{i_2,i_3} \cdots q_{i_m,i_1} = q_{i_1,i_m} \cdot q_{i_m,i_{m-1}} \cdots q_{i_2,i_1}$$

i.e. the product of the transition rates round the cycle are the same in both directions.

One can show that the Kolmogorov criterion is equivalent with the detailed balance conditions and thus gives a necessary and sufficient condition for the reversibility of the process.

Corollary: Since a tree-structured transition diagram does not have any cycles, the Kolmogorov criterion is always satisfied and it follows again that the Markov process represented by a tree is time reversible.

Burke's theorem

The result known as Burke's theorem states:

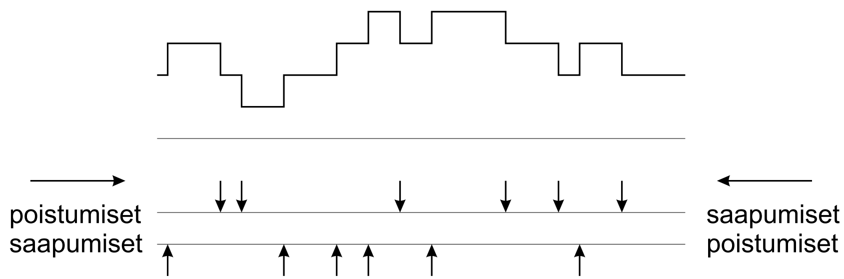
In an $M/M/1$ system, with Poisson arrival rate λ,

a) the departure instants of the customers constitute a Poisson process with intensity λ,

b) for all t the number of customers in the system, N_t, is independent of the output process before time t.

Proof.

a) The $M/M/1$ queue is time reversible. The reversed system behaves exactly as an $M/M/1$ queue. The departure process of the original queue is the same as the arrival process of the reversed system, which being identical with the arrival process of the original system is a Poisson process with intensity λ.



b) The departure epochs of the reversed process before time t are arrival epochs of the original system after time t. Since the arrival instants constitute a Poisson process, its development after time t is independent of anything that has happended before t, and, in particular, of the current value of N_t.

Burke's theorem (continued)

Corollary 1

By observing the departure process of an $M/M/1$ queue one cannot conclude anything about the current number of customers in the system.

- If one has observed a burst in the output process, this is an indication that in the queue there probably has been more customers than usual, but one does not get any information about the current number in system.
- By observing the output process, it is not either possible to get any information about the average service time $1/\mu$.

Corollary 2

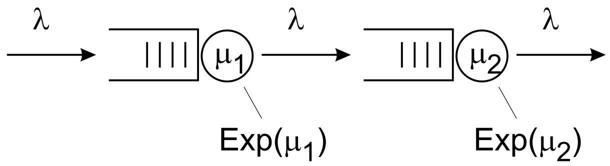
In an open queueing network, which additionally is acyclic (does not contain any feedback loops) all the queues are independent $M/M/1$ queues.

- The arrival processes to each of the queues are indeed Poisson processes by the theorem.
- The current states of the feeding queues are independent of their output processes prior to the current time; the state of the receiving queue, on the other hand depend, only on this prior output process.

Remark

Burke's theorem holds also for the $M/M/m$ and $M/M/\infty$ systems.

Example: Tandem queue



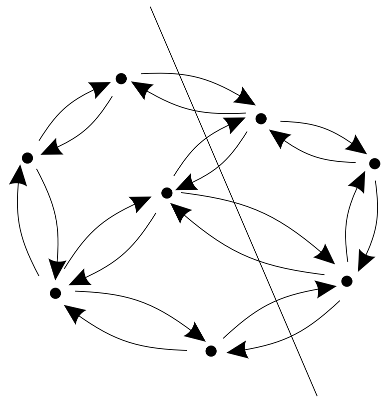
- Independent exponentially distributed service times.
- The first queue is an ordinary $M/M/1$ queue.
- By Burke's theorem its output process is a Poisson process.
- Thus, also queue 2 is an $M/M/1$ queue.
- The state of queue 2, N_2, at time t depends solely on the arrivals before time t.
- By Burke's theorem this arrival process (= the departure process of queue 1) before time t is independent of the state of queue 1 at time t.

$\Rightarrow$ N_1 and N_2 are independent

$$\begin{cases} \mathrm{P}\{N_1 = i\} = (1-\rho_1)\rho_1^i, & \rho_1 = \lambda/\mu_1 \\ \mathrm{P}\{N_2 = j\} = (1-\rho_2)\rho_2^j, & \rho_2 = \lambda/\mu_2 \end{cases}$$

$$\boxed{\mathrm{P}\{N_1 = i, N_2 = j\} = (1-\rho_1)(1-\rho_2)\rho_1^i\rho_2^j}$$

Truncation of a reversible process



Let X_t be a reversible Markov process with the state space $\mathcal{S}$ and with equilibrium probabilities π_i. Reversibility means that the detailed balance conditions hold.

Let $\mathcal{S}'$ be a subset of the state space. Consider the truncated process X'_t, for which

$$q'_{i,j} = \begin{cases} q_{i,j}, & i, j \in \mathcal{S}' \\ 0, & \text{otherwise} \end{cases}$$

Assume further that X'_t is irreducible. Then the process X'_t is reversible and its equilibrium distribution is

$$\boxed{\pi'_i = \frac{\pi_i}{\sum_{j \in \mathcal{S}'} \pi_j}} \qquad \text{ts. } \mathrm{P}\{X' = i\} = \mathrm{P}\{X = i \,|\, X \in \mathcal{S}'\}$$

Proof: Substitute π_i as a trial to the global balance conditions of the process X'_t. One sees immediately that these are satisfied for all states $i \in \mathcal{S}'$ since the net probability flow to any state due to the transitions that have been removed was zero. The equilibrium distribution is obtained by (re)normalizing the distribution over the states $j \in \mathcal{S}'$.

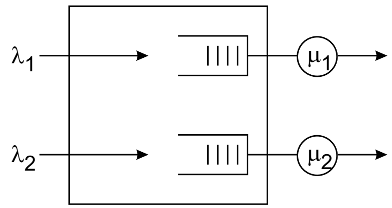
Remark. This truncation principle is very important in practical applications.

Truncation of a reversible process (continued)

Example 1. Before have seen seen that the truncation of a birth-death process (reversible!) with an infinite state space to a finite number of states only implies the truncation and renormalization of the distribution:

$$\underbrace{M/M/1/K \text{ queue}}_{\substack{\text{Truncated geom.}\\ \text{distribution}}}, \quad \underbrace{M/M/m/m}_{\substack{\text{Erlang, truncated}\\ \text{Poisson ditr.}}}, \quad \underbrace{M/M/m/m/n}_{\substack{\text{Engset, truncated}\\ \text{binomial distr.}}}$$

Example 2. Two $M/M/1$ queues which share the common buffer space (waiting room)

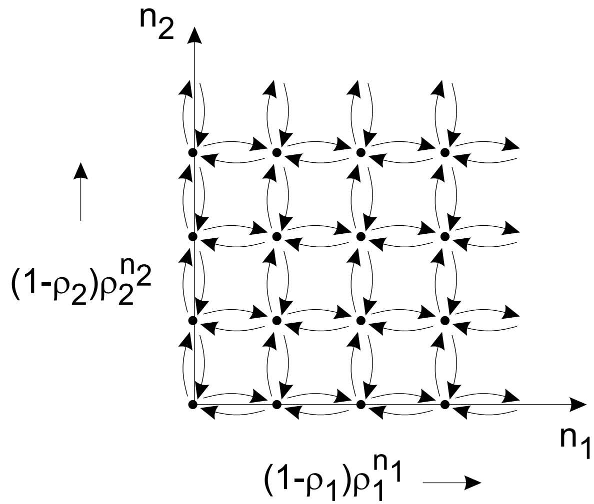


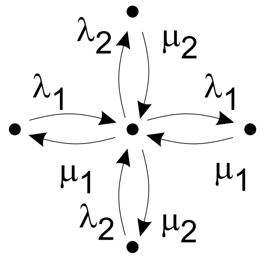
Assume first that the buffer space is infinite. Then the queues are completely independent.

$$\begin{cases} \mathrm{P}\{N_1 = i\} &= (1-\rho_1)\rho_1^i \\ \mathrm{P}\{N_2 = j\} &= (1-\rho_2)\rho_2^j \end{cases}$$

$$\mathrm{P}\{N_1 = i, N_2 = j\} = (1-\rho_1)\rho_1^i(1-\rho_2)\rho_2^j$$

The processes N_1 and N_2 separately are reversible. It is easy to show that then also the joint process (N_1, N_2) is reversible (left as an exercise), i.e. that the above joint distribution satisfies the detailed balance conditions.

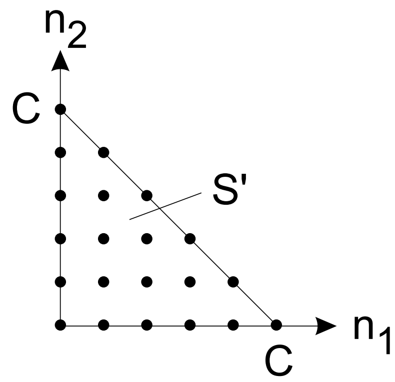




Truncation of a reversible process (continued)

Example 2. continues...

In an infinite capacity system the state probabilities are of product form (product of the marginal distributions of queues 1 and 2)



When the buffer has a finite capacity C, the state space is truncated as shown in the picture.

In the truncated state space $\mathcal{S}'$ the state probabilities are of the same form as before.

$$\mathrm{P}\{N_1 = i, N_2 = j\} = \begin{cases} a \cdot \rho_1^i \cdot \rho_2^j, & i + j \leq C \\ 0, & \text{otherwise} \end{cases}$$

where a is the normalization constant

$$a = \frac{1}{\sum\limits_{\substack{i \\ i+j \leq C}} \sum\limits_{j} \rho_1^i \cdot \rho_2^j}$$

Note. Although the solution is of product form in $\mathcal{S}'$, it is not of that form for all $i, j = 0, 1, 2, \ldots$. The queues are no longer independent but depend via the capacity constraint.

The use of the reversed process for the solution of a queueing problem

Example. Finished work in an $M/M/1$ queue

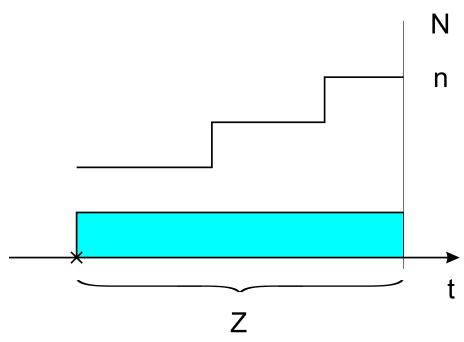
In an $M/M/1$ queue the unfinished time of a customer in service is, due the memoryless property of the exponential distribution, distributed as Exp(μ) and independent of the queue length.

The finished work Z of a customer, in contrast, correlates with the queue length: if the service has taken a long time, it is likely that a long queue has been formed.

By a time reversal argument one can easily derive the conditional distribution of Z given that $N = n$.

The picture shows the development of the queue length

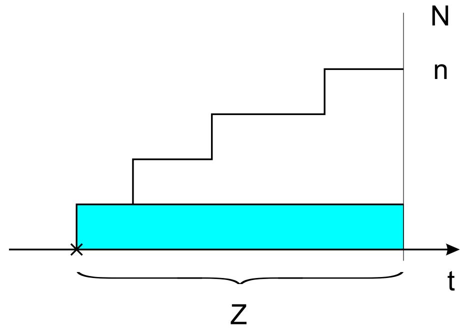
- Starting from the instant (time 0) when the customer being served entered the server, N_t is a non-decreasing function of time; since the service continues there have been no departures from the system.



Example. Finished work in an $M/M/1$ queue (continued)

There are two cases with regard to the arrival instant:

a) The customer arrived at an empty system.



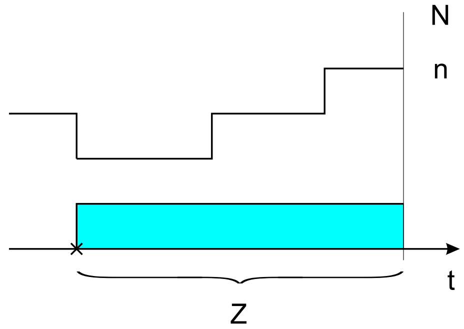
In the reversed time the instant Z is the time when all n customers in the system have departed

$$Z \sim X_1 + X_2 + \cdots + X_n$$

$$X_i \sim \text{Exp}(\mu), \quad i = 1, 2, \ldots, n$$

$$\Rightarrow Z \sim \text{Erlang}(n, \mu)$$

b) The customer arrived at a queue: the instant of the start of the service $-Z$ is the time when the previous customer departed from the system.



In the reversed time, the time Z is the arrival instant of the first customer

$$Z \sim \text{Exp}(\lambda)$$

In the reversed time, starting from the initial state $N = n$, departures occur from the system at intervals distributed as $\text{Exp}(\mu)$ and arrivals with $\text{Exp}(\lambda)$ distributed intervals. Either the system first empties or there is a new arrival to the system. Which one of these two events occurs first determines the instant Z:

$$Z \sim \min(X, Y), \quad X \sim \text{Erlang}(n, \mu) \quad Y \sim \text{Exp}(\lambda); \qquad X \text{ and } Y \text{ are independent}$$