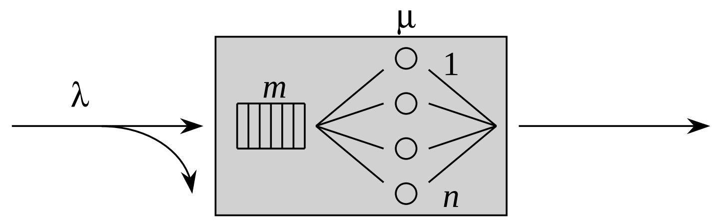
8. Queueing systems

Contents

- Refresher: Simple teletraffic model
- M/M/1 (1 server, ∞ waiting places)
- M/M/n (n servers, ∞ waiting places)

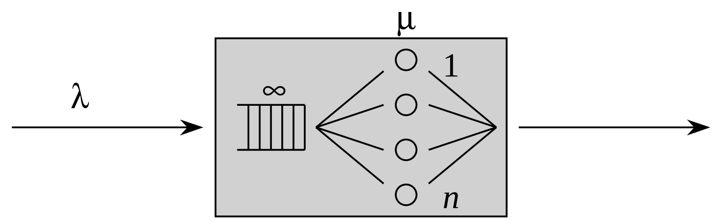
Simple teletraffic model

- **Customers arrive** at rate λ (customers per time unit)
  - $1/\lambda$ = average inter-arrival time
- Customers are **served** by n parallel **servers**
- When busy, a server serves at rate μ (customers per time unit)
  - $1/\mu$ = average service time of a customer
- There are m **waiting** places



Pure waiting system

- Infinite number of waiting places ($m = \infty$)
  - If all n servers are occupied when a customer arrives, she occupies one of the waiting places
  - No customers are lost but some of them have to wait before getting served
- From the customer's point of view, it is interesting to know e.g.
  - what is the probability that she has to wait "too long"?



Queueing discipline

- Consider a single server ($n = 1$) queueing system
- **Queueing discipline** determines the way the server serves the customers
  - It tells
    - whether the customers are served one-by-one or simultaneously
  - Furthermore, if the customers are served one-by-one, it tells
    - in which order they are taken into the service
  - And if the customers are served simultaneously, it tells
    - how the service capacity is shared among them
- A queueing discipline is called **work-conserving** if customers are served with full service rate μ whenever the system is non-empty

Various work-conserving queueing disciplines

- First In First Out (FIFO) = First Come First Served (FCFS)
  - the most ordinary queueing discipline (“queue”)
  - customers served one-by-one (with full service rate μ)
  - always serve the customer that has been waiting for the longest time
- Last In First Out (LIFO) = Last Come First Served (LCFS)
  - “stack”
  - customers served one-by-one (with full service rate μ)
  - always serve the customer that has been waiting for the shortest time
- Processor Sharing (PS)
  - “fair queueing”
  - customers served simultaneously
  - when i customers in the system, each of them served with equal rate μ/i

Contents

M/M/1 queue

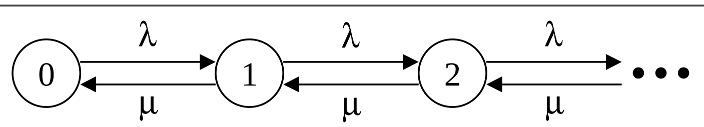
- Consider the following simple teletraffic model:
  - Infinite number of independent customers ($k = \infty$)
  - Interarrival times are IID and exponentially distributed with mean $1/\lambda$
    - so, customers arrive according to a Poisson process with intensity λ
  - One server ($n = 1$)
  - Service times are IID and exponentially distributed with mean $1/\mu$
  - Infinite number of waiting places ($m = \infty$)
  - Default queueing discipline: FIFO
- Using Kendall's notation, this is an **M/M/1 queue**
  - more precisely: M/M/1-FIFO queue
- Notation:
  - $\rho = \lambda/\mu$ = traffic load

Interesting random variables

- X = number of customers in the system at an arbitrary time
  = queue length in equilibrium
- X^* = number of customers in the system at an (typical) arrival time
  = queue length seen by an arriving customer
- W = waiting time of a (typical) customer
- S = service time of a (typical) customer
- $D = W + S$ = total time in the system of a (typical) customer = delay

State transition diagram

- Let $X(t)$ denote the number of customers in the system at time t
  - Assume that $X(t) = i$ at some time t, and
    consider what happens during a short time interval $(t, t+h]$:
    - with prob. $\lambda h + o(h)$,
      a new customer arrives (state transition $i \to i+1$)
    - if $i > 0$, then, with prob. $\mu h + o(h)$,
      a customer leaves the system (state transition $i \to i-1$)
- Process $X(t)$ is clearly a Markov process with state transition diagram



- Note that process $X(t)$ is an irreducible birth-death process
  with an infinite state space $S = \{0,1,2,...\}$

Equilibrium distribution (1)

- Local balance equations (LBE):

$$\pi_i \lambda = \pi_{i+1} \mu \qquad \text{(LBE)}$$

$$\Rightarrow \quad \pi_{i+1} = \frac{\lambda}{\mu} \pi_i = \rho \pi_i$$

$$\Rightarrow \quad \pi_i = \rho^i \pi_0, \quad i = 0,1,2,\ldots$$

- Normalizing condition (N):

$$\sum_{i=0}^{\infty} \pi_i = \pi_0 \sum_{i=0}^{\infty} \rho^i = 1 \qquad \text{(N)}$$

$$\Rightarrow \quad \pi_0 = \left(\sum_{i=0}^{\infty} \rho^i \right)^{-1} = \left(\frac{1}{1-\rho} \right)^{-1} = 1 - \rho, \quad \text{if } \rho < 1$$

Equilibrium distribution (2)

- Thus, for a **stable** system (ρ < 1), the equilibrium distribution exists and is a **geometric distribution**:

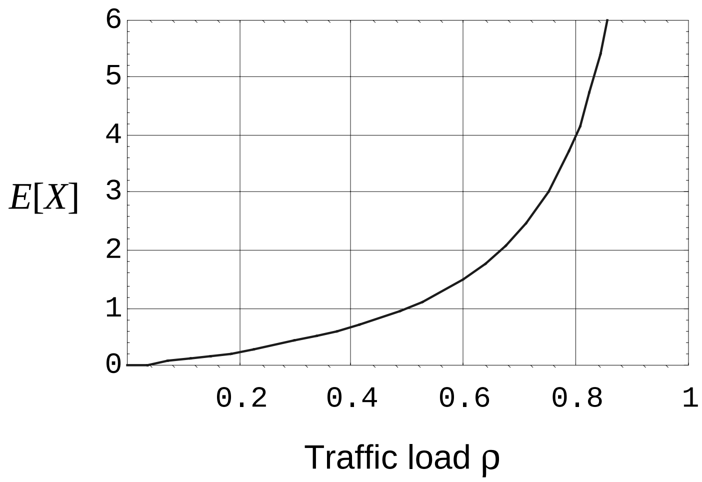
$$\rho < 1 \quad \Rightarrow \quad X \sim \mathrm{Geom}(\rho)$$

$$P\{X = i\} = \pi_i = (1-\rho)\rho^i, \quad i = 0,1,2,\ldots$$

$$E[X] = \frac{\rho}{1-\rho}, \quad D^2[X] = \frac{\rho}{(1-\rho)^2}$$

- Remarks:
  - This result is valid for any **work-conserving** queueing discipline
    - FIFO, LIFO, PS, ...
  - This result is **not** insensitive to the service time distribution as far as the FIFO queueing discipline is concerned
  - However, for any **symmetric** queueing discipline (such as LIFO or PS) the result is, indeed, insensitive to the service time distribution

Mean queue length $E[X]$ vs. traffic load ρ



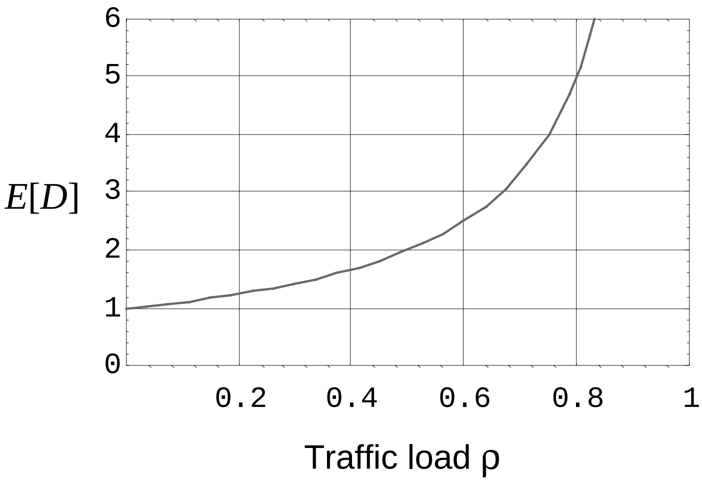
Mean delay

- Let D denote the total time (delay) in the system of a (typical) customer
  - including both the waiting time W and the service time S: $D = W + S$
- Little's formula: $E[X] = \lambda \cdot E[D]$. Thus,

$$E[D] = \frac{E[X]}{\lambda} = \frac{1}{\lambda} \cdot \frac{\rho}{1-\rho} = \frac{1}{\mu} \cdot \frac{1}{1-\rho} = \frac{1}{\mu - \lambda}$$

- Remarks:
  - The mean delay is the same for all **work-conserving** queueing disciplines
    - FIFO, LIFO, PS, ...
  - But the variance and other moments are different!

Mean delay $E[D]$ vs. traffic load ρ



Mean waiting time

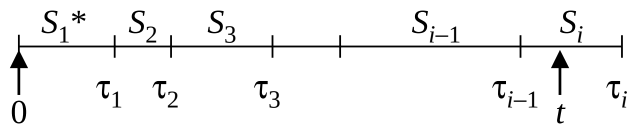
- Let W denote the waiting time of a (typical) customer
- Since $W = D - S$, we have

$$E[W] = E[D] - E[S] = \frac{1}{\mu} \cdot \frac{1}{1-\rho} - \frac{1}{\mu} = \frac{1}{\mu} \cdot \frac{\rho}{1-\rho}$$

- Remarks:
  - The mean waiting time is the same for all **work-conserving** queueing disciplines
    - FIFO, LIFO, PS, ...
  - But the variance and other moments are different!

Waiting time distribution (1)

- Let W denote the waiting time of a (typical) customer
- Let X^* denote the number of customers in the system at the arrival time
- PASTA: $P\{X^* = i\} = P\{X = i\} = \pi_i$.
- Assume now, for a while, that $X^* = i$
  - Service times $S_2,\ldots,S_i$ of the waiting customers are IID and $\sim \mathrm{Exp}(\mu)$
  - Due to the memoryless property of the exponential distribution, the **remaining** service time S_1^* of the customer in service also follows $\mathrm{Exp}(\mu)$-distribution (and is independent of everything else)
  - Due to the FIFO queueing discipline, $W = S_1^* + S_2 + \ldots + S_i$
  - Construct a Poisson (point) process τ_n by defining $\tau_1 = S_1^*$ and $\tau_n = S_1^* + S_2 + \ldots + S_n$, $n \geq 2$. Now (since $X^* = i$): $W > t \Leftrightarrow \tau_i > t$



Waiting time distribution (2)

- Since $W = 0 \Leftrightarrow X^* = 0$, we have

$$P\{W = 0\} = P\{X^* = 0\} = \pi_0 = 1 - \rho$$

$$P\{W > t\} = \sum_{i=1}^{\infty} P\{W > t \mid X^* = i\} P\{X^* = i\}$$

$$= \sum_{i=1}^{\infty} P\{\tau_i > t\} \pi_i = \sum_{i=1}^{\infty} P\{\tau_i > t\}(1-\rho)\rho^i$$

- Denote by $A(t)$ the Poisson (counter) process corresponding to τ_n
  - It follows that: $\tau_i > t \Leftrightarrow A(t) \leq i$-1
  - On the other hand, we know that $A(t) \sim \mathrm{Poisson}\,(\mu t)$. Thus,

$$P\{\tau_i > t\} = P\{A(t) \leq i-1\} = \sum_{j=0}^{i-1} \frac{(\mu t)^j}{j!} e^{-\mu t}$$

Waiting time distribution (3)

- By combining the previous formulas, we get

$$P\{W > t\} = \sum_{i=1}^{\infty} P\{\tau_i > t\}(1-\rho)\rho^i$$
$$= \sum_{i=1}^{\infty}\sum_{j=0}^{i-1} \frac{(\mu t)^j}{j!} e^{-\mu t}(1-\rho)\rho^i$$
$$= \rho \sum_{j=0}^{\infty} \frac{(\mu t\rho)^j}{j!} e^{-\mu t}(1-\rho) \sum_{i=j+1}^{\infty} \rho^{i-(j+1)}$$
$$= \rho \sum_{j=0}^{\infty} \frac{(\mu t\rho)^j}{j!} e^{-\mu t} = \rho\, e^{\mu t\rho} e^{-\mu t} = \rho\, e^{-\mu(1-\rho)t}$$

Waiting time distribution (4)

- Waiting time W can thus be presented as a product $W = JD$ of two independent random variables $J \sim \mathrm{Bernoulli}(\rho)$ and $D \sim \mathrm{Exp}(\mu(1-\rho))$:

$$P\{W = 0\} = P\{J = 0\} = 1-\rho$$
$$P\{W > t\} = P\{J = 1, D > t\} = \rho \cdot e^{-\mu(1-\rho)t}, \quad t > 0$$
$$E[W] = E[J]E[D] = \rho \cdot \frac{1}{\mu(1-\rho)} = \frac{1}{\mu} \cdot \frac{\rho}{1-\rho}$$
$$E[W^2] = P\{J = 1\}E[D^2] = \rho \cdot \frac{2}{\mu^2(1-\rho)^2} = \frac{1}{\mu^2} \cdot \frac{2\rho}{(1-\rho)^2}$$
$$D^2[W] = E[W^2] - E[W]^2 = \frac{1}{\mu^2} \cdot \frac{\rho(2-\rho)}{(1-\rho)^2}$$

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M/M/*n* queue

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  - Finite number of servers ($n < \infty$)
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  - Infinite number of waiting places ($m = \infty$)
  - Default queueing discipline: FCFS
- Using Kendall's notation, this is an **M/M/*n* queue**
  - more precisely: M/M/n-FCFS queue
- Notation:
  - $\rho = \lambda/(n\mu)$ = traffic load

State transition diagram

- Let $X(t)$ denote the number of customers in the system at time t
  - Assume that $X(t) = i$ at some time t, and consider what happens during a short time interval $(t, t+h]$:
    - with prob. $\lambda h + o(h)$, a new customer arrives (state transition $i \to i+1$)
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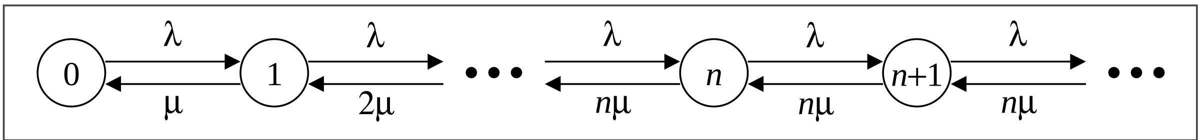


- Note that process $X(t)$ is an irreducible birth-death process with an infinite state space $S = \{0,1,2,...\}$

Equilibrium distribution (1)

- Local balance equations (LBE) for $i < n$:

$$\pi_i \lambda = \pi_{i+1}(i+1)\mu \qquad \text{(LBE)}$$

$$\Rightarrow \quad \pi_{i+1} = \frac{\lambda}{(i+1)\mu}\pi_i = \frac{n\rho}{i+1}\pi_i$$

$$\Rightarrow \quad \pi_i = \frac{(n\rho)^i}{i!}\pi_0, \quad i = 0,1,\ldots,n$$

- Local balance equations (LBE) for $i \geq n$:

$$\pi_i \lambda = \pi_{i+1} n\mu \qquad \text{(LBE)}$$

$$\Rightarrow \quad \pi_{i+1} = \frac{\lambda}{n\mu}\pi_i = \rho\pi_i$$

$$\Rightarrow \quad \pi_i = \rho^{i-n}\pi_n = \rho^{i-n}\frac{(n\rho)^n}{n!}\pi_0 = \frac{n^n\rho^i}{n!}\pi_0, \quad i = n, n+1, \ldots$$

Equilibrium distribution (2)

- Normalizing condition (N):

$$\sum_{i=0}^{\infty}\pi_i=\pi_0\left(\sum_{i=0}^{n-1}\frac{(n\rho)^i}{i!}+\sum_{i=n}^{\infty}\frac{n^n\rho^i}{n!}\right)=1 \qquad \text{(N)}$$

$$\Rightarrow\quad \pi_0=\left(\sum_{i=0}^{n-1}\frac{(n\rho)^i}{i!}+\frac{(n\rho)^n}{n!}\sum_{i=n}^{\infty}\rho^{i-n}\right)^{-1}$$

$$=\left(\sum_{i=0}^{n-1}\frac{(n\rho)^i}{i!}+\frac{(n\rho)^n}{n!(1-\rho)}\right)^{-1}=\frac{1}{\alpha+\beta},\quad \text{if } \rho<1$$

$$\text{Notation}:\quad \alpha=\sum_{i=0}^{n-1}\frac{(n\rho)^i}{i!},\quad \beta=\frac{(n\rho)^n}{n!(1-\rho)}$$

Equilibrium distribution (3)

- Thus, for a **stable** system (ρ < 1, that is: λ < *n*μ), the equilibrium distribution exists and is as follows:

$$\rho<1\ \Rightarrow$$

$$P\{X=i\}=\pi_i=\begin{cases}\frac{(n\rho)^i}{i!}\cdot\frac{1}{\alpha+\beta}, & i=0,1,\ldots,n\\ \frac{n^n\rho^i}{n!}\cdot\frac{1}{\alpha+\beta}, & i=n,n+1,\ldots\end{cases}$$

$$n=1:\ \alpha=1,\ \beta=\frac{\rho}{1-\rho},\ \pi_0=\frac{1}{\alpha+\beta}=1-\rho$$

$$n=2:\ \alpha=1+2\rho,\ \beta=\frac{2\rho^2}{1-\rho},\ \pi_0=\frac{1}{\alpha+\beta}=\frac{1-\rho}{1+\rho}$$

Probability of waiting

- Let p_W denote the probability that an arriving customer has to wait
- Let X^* denote the number of customers in the system at an arrival time
- An arriving customer has to wait whenever all the servers are occupied at her arrival time. Thus,

$$p_W = P\{X^* \geq n\}$$

- PASTA: $P\{X^* = i\} = P\{X = i\} = \pi_i$. Thus,

$$p_W = P\{X^* \geq n\} = \sum_{i=n}^{\infty} \pi_i = \sum_{i=n}^{\infty} \pi_0 \cdot \frac{n^n \rho^i}{n!} = \pi_0 \cdot \frac{(n\rho)^n}{n!(1-\rho)} = \frac{\beta}{\alpha+\beta}$$

$$n=1: \quad p_W = \rho$$

$$n=2: \quad p_W = \frac{2\rho^2}{1+\rho}$$

Mean number of waiting customers

- Let X_W denote the number of waiting customers in equilibrium
- Then

$$E[X_W] = \sum_{i=n}^{\infty} (i-n)\pi_i = \pi_0 \frac{(n\rho)^n}{n!(1-\rho)} \sum_{i=n}^{\infty} (i-n) \cdot (1-\rho)\rho^{i-n}$$

$$= p_W \cdot \frac{\rho}{1-\rho}$$

$$n=1: \quad E[X_W] = p_W \cdot \frac{\rho}{1-\rho} = \frac{\rho^2}{1-\rho}$$

$$n=2: \quad E[X_W] = p_W \cdot \frac{\rho}{1-\rho} = \frac{2\rho^2}{1+\rho} \cdot \frac{\rho}{1-\rho} = \frac{2\rho^3}{1-\rho^2}$$

Mean waiting time

- Let W denote the waiting time of a (typical) customer
- Little's formula: $E[X_W] = \lambda \cdot E[W]$. Thus,

$$E[W] = \frac{E[X_W]}{\lambda} = \frac{1}{\lambda} \cdot p_W \cdot \frac{\rho}{1-\rho} = \frac{1}{\mu} \cdot \frac{p_W}{n(1-\rho)} = p_W \cdot \frac{1}{n\mu - \lambda}$$

$$n = 1: \quad E[W] = \frac{1}{\mu} \cdot \frac{p_W}{1-\rho} = \frac{1}{\mu} \cdot \frac{\rho}{1-\rho}$$

$$n = 2: \quad E[W] = \frac{1}{\mu} \cdot \frac{p_W}{2(1-\rho)} = \frac{1}{\mu} \cdot \frac{\rho^2}{1-\rho^2}$$

Mean delay

- Let D denote the total time (delay) in the system of a (typical) customer
  - including both the waiting time W and the service time S: $D = W + S$
- Then,

$$E[D] = E[W] + E[S] = \frac{1}{\mu} \cdot \left(\frac{p_W}{n(1-\rho)} + 1 \right) = p_W \cdot \frac{1}{n\mu - \lambda} + \frac{1}{\mu}$$

$$n = 1: \quad E[D] = \frac{1}{\mu} \cdot \left(\frac{p_W}{1-\rho} + 1 \right) = \frac{1}{\mu} \cdot \left(\frac{\rho}{1-\rho} + 1 \right) = \frac{1}{\mu} \cdot \frac{1}{1-\rho}$$

$$n = 2: \quad E[D] = \frac{1}{\mu} \cdot \frac{p_W}{2(1-\rho)} = \frac{1}{\mu} \cdot \left(\frac{\rho^2}{1-\rho^2} + 1 \right) = \frac{1}{\mu} \cdot \frac{1}{1-\rho^2}$$

Mean queue length

- Let X denote the number of customers in the system (queue length) in equilibrium
- Little's formula: $E[X] = \lambda \cdot E[D]$. Thus,

$$E[X] = \lambda \cdot E[D] = p_W \cdot \frac{\lambda}{n\mu - \lambda} + \frac{\lambda}{\mu} = p_W \cdot \frac{\rho}{1-\rho} + n\rho$$

$$n = 1: \quad E[X] = p_W \cdot \frac{\rho}{1-\rho} + \rho = \rho \cdot \frac{\rho}{1-\rho} + \rho = \frac{\rho}{1-\rho}$$

$$n = 2: \quad E[X] = p_W \cdot \frac{\rho}{1-\rho} + 2\rho = \frac{2\rho^2}{1+\rho} \cdot \frac{\rho}{1-\rho} + 2\rho = \frac{2\rho}{1-\rho^2}$$

Waiting time distribution (1)

- Let W denote the waiting time of a (typical) customer
- Let X^* denote the number of customers in the system at the arrival time
- The customer has to wait only if $X^* \geq n$. This happens with prob. p_W.
- Under the assumption that $X^* = i \geq n$, the system, however, looks like an ordinary M/M/1 queue with arrival rate λ and service rate $n\mu$.
  - Let W' denote the waiting time of a (typical) customer in this M/M/1 queue
  - Let $X^{*'}$ denote the number of customers in the system at the arrival time
- It follows that

$$P\{W = 0\} = 1 - p_W$$

$$P\{W > t\} = P\{X^* \geq n\} P\{W > t \mid X^* \geq n\}$$

$$= p_W \cdot P\{W' > t \mid X^{*'} \geq 1\} = p_W \cdot e^{-n\mu(1-\rho)t}, \quad t > 0$$

Waiting time distribution (2)

- Waiting time W can thus be presented as a product $W = JD'$ of two indep. random variables $J \sim \text{Bernoulli}(p_W)$ and $D' \sim \text{Exp}(n\mu(1-\rho))$:

$$P\{W = 0\} = P\{J = 0\} = 1 - p_W$$

$$P\{W > t\} = P\{J = 1, D' > t\} = p_W \cdot e^{-n\mu(1-\rho)t}, \quad t > 0$$

$$E[W] = E[J]E[D'] = p_W \cdot \frac{1}{n\mu(1-\rho)} = \frac{1}{\mu} \cdot \frac{p_W}{n(1-\rho)}$$

$$E[W^2] = P\{J = 1\}E[D'^2] = p_W \cdot \frac{2}{n^2\mu^2(1-\rho)^2} = \frac{1}{\mu^2} \cdot \frac{2p_W}{n^2(1-\rho)^2}$$

$$D^2[W] = E[W^2] - E[W]^2 = \frac{1}{\mu^2} \cdot \frac{p_W(2-p_W)}{n^2(1-\rho)^2}$$

Example (1)

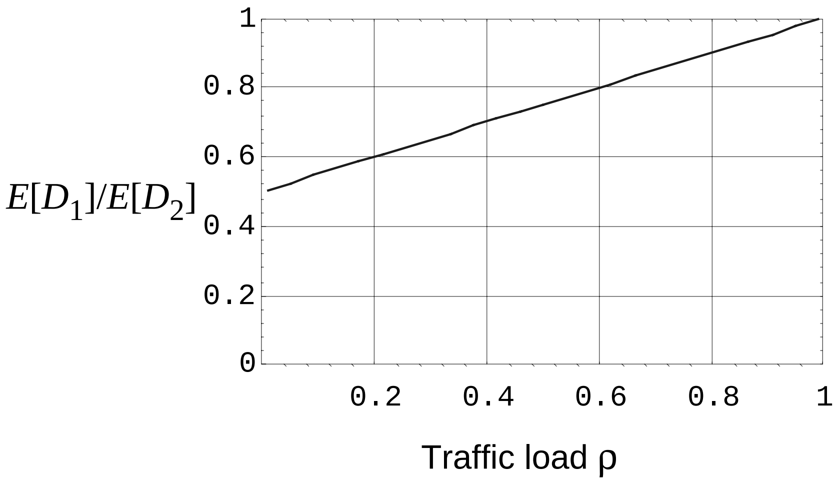
- Printer problem
  - Consider the following two different configurations:
    - One rapid printer (IID printing times ~ $\text{Exp}(2\mu)$)
    - Two slower parallel printers (IID printing times ~ $\text{Exp}(\mu)$)
  - Criterion: minimize mean delay $E[D]$
    - One rapid printer (M/M/1 model with $\rho = \lambda/(2\mu)$):

$$E[D_1] = \frac{1}{2\mu} \cdot \frac{1}{1-\rho}$$

- Two slower printers (M/M/2 model with $\rho = \lambda/(2\mu)$):

$$E[D_2] = \frac{1}{\mu} \cdot \frac{1}{1-\rho^2} = \frac{1}{2\mu} \cdot \frac{2}{(1-\rho)(1+\rho)} = E[D_1] \cdot \frac{2}{1+\rho} > E[D_1]$$

Example (2)



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