9. Simulation (supplement)

On confidence intervals (1)

- This is a supplement to Slide 43 of Lecture 9!
- Assume that X_i's are IID with unknown mean α and **known** variance σ^2
- By the Central Limit Theorem (see Lecture 5, Slide 49), for large n,

$$Z := \frac{\overline{X}_n - \alpha}{\sigma / \sqrt{n}} \approx \mathrm{N}(0,1)$$

- **Definition**: The interval ($\overline{X}_n - y$, $\overline{X}_n + y$) is called the **confidence interval** of the sample mean at **confidence level** $1 - \beta$ if

$$P\{|\overline{X}_n - \alpha| \le y\} = 1 - \beta$$

- Note: “with probability $1 - \beta$, the parameter α belongs to this interval”

On confidence intervals (2)

- Let z_p denote the p-fractile of the N(0,1) distribution, that is: $P\{Z \le z_p\} = p$, where $Z \sim N(0,1)$
  - Example: for $\beta = 5\%$, $z_{1-(\beta/2)} = z_{0.975} \approx 1.96 \approx 2$
- **Proposition**: The confidence interval for the sample average at confidence level $1 - \beta$ is

$$\bar{X}_n \pm z_{1-\frac{\beta}{2}} \cdot \frac{\sigma}{\sqrt{n}}$$

- **Proof**: By the definition, we have to show that

$$P\{|\bar{X}_n - \alpha| \le z_{1-\frac{\beta}{2}} \cdot \frac{\sigma}{\sqrt{n}}\} = 1 - \beta$$

$$P\{|\bar{X}_n - \alpha| \le y\} = 1 - \beta$$

$$\Leftrightarrow P\{\frac{|\bar{X}_n - \alpha|}{\sigma/\sqrt{n}} \le \frac{y}{\sigma/\sqrt{n}}\} = 1 - \beta$$

$$\Leftrightarrow P\{\frac{-y}{\sigma/\sqrt{n}} \le \frac{\bar{X}_n - \alpha}{\sigma/\sqrt{n}} \le \frac{y}{\sigma/\sqrt{n}}\} = 1 - \beta$$

$$\Leftrightarrow \Phi(\frac{y}{\sigma/\sqrt{n}}) - \Phi(\frac{-y}{\sigma/\sqrt{n}}) = 1 - \beta \qquad [\Phi(x) := P\{Z \le x\}]$$

$$\Leftrightarrow \Phi(\frac{y}{\sigma/\sqrt{n}}) - (1 - \Phi(\frac{y}{\sigma/\sqrt{n}})) = 1 - \beta \qquad [\Phi(-x) = 1 - \Phi(x)]$$

$$\Leftrightarrow \Phi(\frac{y}{\sigma/\sqrt{n}}) = 1 - \frac{\beta}{2}$$

$$\Leftrightarrow \frac{y}{\sigma/\sqrt{n}} = z_{1-\frac{\beta}{2}}$$

$$\Leftrightarrow y = z_{1-\frac{\beta}{2}} \cdot \frac{\sigma}{\sqrt{n}}$$