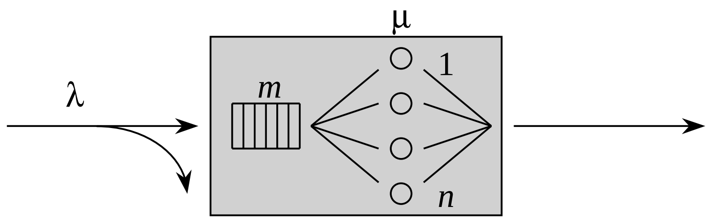
7. Loss systems

Contents

- Refresher: Simple teletraffic system
- Poisson model (∞ customers, ∞ servers)
- Erlang model (∞ customers, $n < \infty$ servers)
- Binomial model ($k < \infty$ customers, $n = k$ servers)
- Engset model ($k < \infty$ customers, $n < k$ servers)

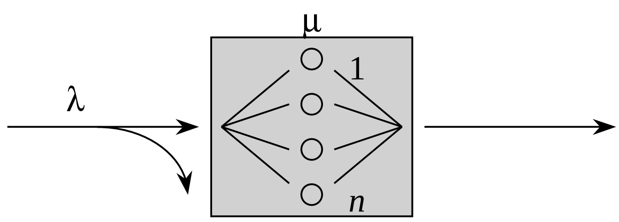
Simple teletraffic model

- **Customers arrive** at rate λ (customers per time unit)
  - $1/\lambda$ = average inter-arrival time
- Customers are **served** by n parallel **servers**
- When busy, a server serves at rate μ (customers per time unit)
  - $1/\mu$ = average service time of a customer
- There are m **waiting** places
- It is assumed that blocked customers (arriving in a full system) are lost



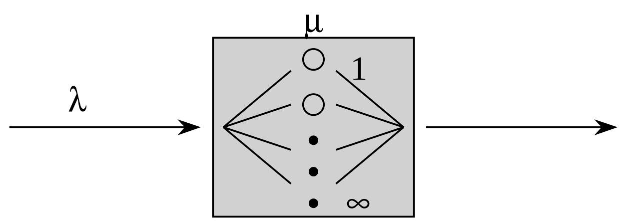
Pure loss system

- No waiting places ($m = 0$)
  - If the system is full (with all n servers occupied) when a customer arrives, she is not served at all but lost
  - Some customers are lost
- From the customer's point of view,
  - it is interesting to know e.g. the blocking probability
- Note: In addition to the case where the arrival rate λ is constant, we will consider the case where it depends on the state of the system: $\lambda(x)$



Infinite system

- Infinite number of servers ($n = \infty$)
  - No customers are lost or even have to wait before getting served
- Note: Also here, in addition to the case where the arrival rate λ is constant, we will consider the case where it depends on the state of the system: $\lambda(x)$



Blocking

- In a loss system some calls are lost
  - a call is lost if all n channels are occupied when the call arrives
  - the term **blocking** refers to this event
- There are (at least) two different types of blocking quantities:
  - **Call blocking** B_c = probability that an arriving call finds all n channels occupied = the fraction of calls that are lost
  - **Time blocking** B_t = probability that all n channels are occupied at an arbitrary time = the fraction of time that all n channels are occupied
- The two blocking quantities are not necessarily equal
  - If calls arrive according to a Poisson process, then $B_c = B_t$
- Call blocking is a better measure for the quality of service experienced by the subscribers but, typically, time blocking is easier to calculate

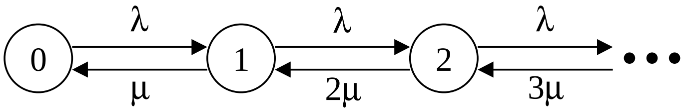
Contents

Poisson model (M/M/∞)

- **Definition**: **Poisson model** is the following simple teletraffic model:
  - Infinite number of independent customers ($k = \infty$)
  - Interarrival times are IID and exponentially distributed with mean $1/\lambda > 0$
    - so, customers arrive according to a Poisson process with intensity λ
  - Infinite number of servers ($n = \infty$)
  - Service times are IID and exponentially distributed with mean $1/\mu > 0$
  - No waiting places ($m = 0$)
- Poisson model:
  - Using Kendall's notation, this is an M/M/∞ queue
  - Infinite system, and, thus, **lossless**
- Notation:
  - $a = \lambda/\mu$ = traffic intensity

State transition diagram

- Let $X(t)$ denote the number of customers in the system at time t
  - Assume that $X(t) = i$ at some time t, and consider what happens during a short time interval $(t, t+h]$:
    - with prob. $\lambda h + o(h)$, a new customer arrives (state transition $i \to i+1$)
    - if $i > 0$, then, with prob. $i\mu h + o(h)$, a customer leaves the system (state transition $i \to i$-1)
- Process $X(t)$ is clearly a Markov process with state transition diagram



- Note that process $X(t)$ is an irreducible birth-death process with an infinite state space $S = \{0,1,2,...\}$

Equilibrium distribution (1)

- Local balance equations (LBE):

$$\pi_i \lambda = \pi_{i+1}(i+1)\mu \qquad \text{(LBE)}$$

$$\Rightarrow \quad \pi_{i+1} = \frac{\lambda}{(i+1)\mu}\pi_i = \frac{a}{i+1}\pi_i$$

$$\Rightarrow \quad \pi_i = \frac{a^i}{i!}\pi_0, \quad i = 0,1,2,\ldots$$

- Normalizing condition (N):

$$\sum_{i=0}^{\infty}\pi_i = \pi_0 \sum_{i=0}^{\infty}\frac{a^i}{i!} = 1 \qquad \text{(N)}$$

$$\Rightarrow \quad \pi_0 = \left(\sum_{i=0}^{\infty}\frac{a^i}{i!}\right)^{-1} = \left(e^a\right)^{-1} = e^{-a}$$

Equilibrium distribution (2)

- Thus, the equilibrium distribution is a **Poisson distribution**:

$$X \sim \text{Poisson}(a)$$

$$P\{X = i\} = \pi_i = \frac{a^i}{i!} e^{-a}, \quad i = 0,1,2,\ldots$$

$$E[X] = a, \quad D^2[X] = a$$

- Remark (insensitivity):
  - The result is **insensitive** to the service time distribution, that is: it is valid for **any** service time distribution with mean $1/\mu$
  - So, instead of the M/M/∞ model, we can consider, as well, the more general M/G/∞ model

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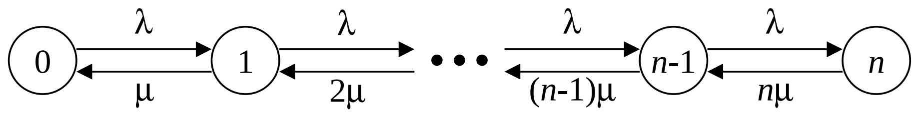
Erlang model (M/M/n/n)

- **Definition**: **Erlang model** is the following simple teletraffic model:
  - Infinite number of independent customers ($k = \infty$)
  - Interarrival times are IID and exponentially distributed with mean $1/\lambda > 0$
    - so, customers arrive according to a Poisson process with intensity λ
  - Finite number of servers ($n < \infty$)
  - Service times are IID and exponentially distributed with mean $1/\mu > 0$
  - No waiting places ($m = 0$)
- Erlang model:
  - Using Kendall's notation, this is an M/M/n/n queue
  - Pure loss system, and, thus, **lossy**
- Notation:
  - $a = \lambda/\mu$ = traffic intensity

State transition diagram

- Let $X(t)$ denote the number of customers in the system at time t
  - Assume that $X(t) = i$ at some time t, and
    consider what happens during a short time interval $(t, t+h]$:
    - with prob. $\lambda h + o(h)$,
      a new customer arrives (state transition $i \rightarrow i+1$)
    - with prob. $i\mu h + o(h)$,
      a customer leaves the system (state transition $i \rightarrow i-1$)
- Process $X(t)$ is clearly a Markov process with state transition diagram



- Note that process $X(t)$ is an irreducible birth-death process
  with a finite state space $S = \{0,1,2,\ldots,n\}$

Equilibrium distribution (1)

- Local balance equations (LBE):

$$\pi_i \lambda = \pi_{i+1}(i+1)\mu \qquad \text{(LBE)}$$

$$\Rightarrow \quad \pi_{i+1} = \frac{\lambda}{(i+1)\mu}\pi_i = \frac{a}{i+1}\pi_i$$

$$\Rightarrow \quad \pi_i = \frac{a^i}{i!}\pi_0, \quad i = 0,1,\ldots,n$$

- Normalizing condition (N):

$$\sum_{i=0}^{n}\pi_i = \pi_0 \sum_{i=0}^{n}\frac{a^i}{i!} = 1 \qquad \text{(N)}$$

$$\Rightarrow \quad \pi_0 = \left(\sum_{i=0}^{n}\frac{a^i}{i!}\right)^{-1}$$

Equilibrium distribution (2)

- Thus, the equilibrium distribution is a **truncated Poisson distribution**:

$$P\{X = i\} = \pi_i = \frac{\frac{a^i}{i!}}{\sum_{j=0}^{n}\frac{a^j}{j!}}, \quad i = 0,1,\ldots,n$$

- Remark (insensitivity):
  - The result is **insensitive** to the service time distribution, that is: it is valid for **any** service time distribution with mean $1/\mu$
  - So, instead of the M/M/n/n model, we can consider, as well, the more general M/G/n/n model

Time blocking

- **Time blocking** B_t = probability that all n channels are occupied at an arbitrary time = the fraction of time that all n channels are occupied
- For a stationary Markov process, this equals the probability π_n of the equilibrium distribution π. Thus,

$$B_t := P\{X = n\} = \pi_n = \frac{\frac{a^n}{n!}}{\sum_{j=0}^{n} \frac{a^j}{j!}}$$

Call blocking

- **Call blocking** B_c = probability that an arriving call finds all n channels occupied = the fraction of calls that are lost
- However, due to Poisson arrivals and PASTA property,
  the probability that an arriving call finds all n channels occupied equals the probability that all n channels are occupied at an arbitrary time,
- In other words, call blocking B_c equals time blocking B_t:

$$B_c = B_t = \frac{\frac{a^n}{n!}}{\sum_{j=0}^{n} \frac{a^j}{j!}}$$

- This is **Erlang's blocking formula**

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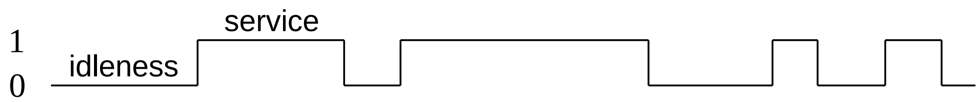
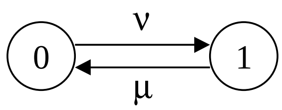
Binomial model (M/M/k/k/k)

- **Definition**: **Binomial model** is the following (simple) teletraffic model:
  - Finite number of independent customers ($k < \infty$)
    - **on-off type** customers (alternating between idleness and activity)
  - Idle times are IID and exponentially distributed with mean $1/\nu > 0$
  - As many servers as customers ($n = k$)
  - Service times are IID and exponentially distributed with mean $1/\mu > 0$
  - No waiting places ($m = 0$)
- Binomial model:
  - Using Kendall's notation, this is an M/M/k/k/k queue
  - Although a finite system, this is clearly **lossless**
- On-off type customer (note: when active, a customer is in service):



On-off type customer (1)

- Let $X_j(t)$ denote the state of customer j ($j = 1,2,\dots,k$) at time t
  - State 0 = idle, state 1 = active = in service
  - Consider what happens during a short time interval $(t, t+h]$:
    - if $X_j(t) = 0$, then, with prob. $\nu h + o(h)$, the customer becomes active (state transition $0 \to 1$)
    - if $X_j(t) = 1$, then, with prob. $\mu h + o(h)$, the customer becomes idle (state transition $1 \to 0$)
- Process $X_j(t)$ is clearly a Markov process with state transition diagram



- Note that process $X_j(t)$ is an irreducible birth-death process with a finite state space $S = \{0,1\}$

On-off type customer (2)

- Local balance equations (LBE):

$$\pi_0^{(j)}\nu = \pi_1^{(j)}\mu \;\Rightarrow\; \pi_1^{(j)} = \frac{\nu}{\mu}\pi_0^{(j)}$$

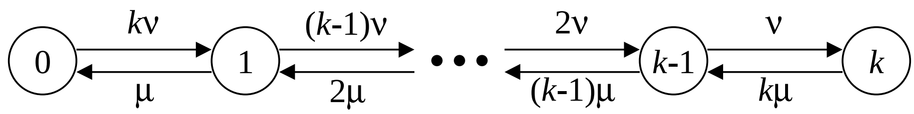
- Normalizing condition (N):

$$\pi_0^{(j)} + \pi_1^{(j)} = \pi_0^{(j)}(1+\frac{\nu}{\mu}) = 1 \;\Rightarrow\; \pi_0^{(j)} = \frac{\mu}{\nu+\mu},\quad \pi_1^{(j)} = \frac{\nu}{\nu+\mu}$$

- So, the equilibrium distribution of a single customer is the **Bernoulli distribution** with success probability $\nu/(\nu+\mu)$
- From this, we could deduce that the equilibrium distribution of the state of the whole system (that is: the number of active customers) is the binomial distribution $\mathrm{Bin}(k, \nu/(\nu+\mu))$

State transition diagram

- Let $X(t)$ denote the number of active customers
  - Assume that $X(t) = i$ at some time t, and consider what happens during a short time interval $(t, t+h]$:
    - if $i < k$, then, with prob. $(k-i)\nu h + o(h)$, an idle customer becomes active (state transition $i \to i+1$)
    - if $i > 0$, then, with prob. $i\mu h + o(h)$, an active customer becomes idle (state transition $i \to i-1$)
- Process $X(t)$ is clearly a Markov process with state transition diagram



- Note that process $X(t)$ is an irreducible birth-death process with a finite state space $S = \{0,1,\ldots,k\}$

Equilibrium distribution (1)

- Local balance equations (LBE):

$$\pi_i (k-i)\nu = \pi_{i+1}(i+1)\mu \qquad \text{(LBE)}$$

$$\Rightarrow \quad \pi_{i+1} = \frac{(k-i)\nu}{(i+1)\mu}\pi_i$$

$$\Rightarrow \quad \pi_i = \frac{k!}{i!(k-i)!}\left(\frac{\nu}{\mu}\right)^i \pi_0 = \binom{k}{i}\left(\frac{\nu}{\mu}\right)^i \pi_0, \quad i = 0,1,\ldots,k$$

- Normalizing condition (N):

$$\sum_{i=0}^{k} \pi_i = \pi_0 \sum_{i=0}^{k} \binom{k}{i}\left(\frac{\nu}{\mu}\right)^i = 1 \qquad \text{(N)}$$

$$\Rightarrow \quad \pi_0 = \left(\sum_{i=0}^{k} \binom{k}{i}\left(\frac{\nu}{\mu}\right)^i\right)^{-1} = \left(1+\frac{\nu}{\mu}\right)^{-k} = \left(\frac{\mu}{\nu+\mu}\right)^k$$

Equilibrium distribution (2)

- Thus, the equilibrium distribution is a **binomial distribution**:

$$X \sim \mathrm{Bin}(k, \tfrac{\nu}{\nu+\mu})$$

$$P\{X = i\} = \pi_i = \binom{k}{i}(\tfrac{\nu}{\nu+\mu})^i (\tfrac{\mu}{\nu+\mu})^{k-i}, \quad i = 0,1,\ldots,k$$

$$E[X] = \frac{k\nu}{\nu+\mu}, \quad D^2[X] = k \cdot \frac{\nu}{\nu+\mu} \cdot \frac{\mu}{\nu+\mu} = \frac{k\nu\mu}{(\nu+\mu)^2}$$

- Remark (insensitivity):
  - The result is **insensitive both** to the service **and** the idle time distribution, that is: it is valid for **any** service time distribution with mean $1/\mu$ and **any** idle time distribution with mean $1/\nu$
  - So, instead of the M/M/k/k/k model, we can consider, as well, the more general G/G/k/k/k model

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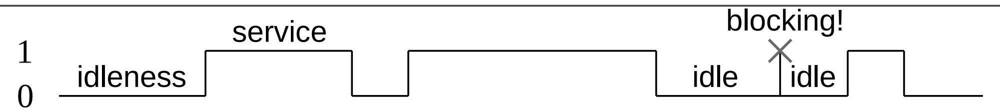
Engset model (M/M/*n*/*n*/*k*)

- **Definition**: **Engset model** is the following (simple) teletraffic model:
  - Finite number of independent customers ($k < \infty$)
    - **on-off type** customers (alternating between idleness and activity)
  - Idle times are IID and exponentially distributed with mean $1/\nu > 0$
  - Less servers than customers ($n < k$)
  - Service times are IID and exponentially distributed with mean $1/\mu > 0$
  - No waiting places ($m = 0$)
- Engset model:
  - Using Kendall's notation, this is an $M/M/n/n/k$ queue
  - This is a pure loss system, and, thus, **lossy**

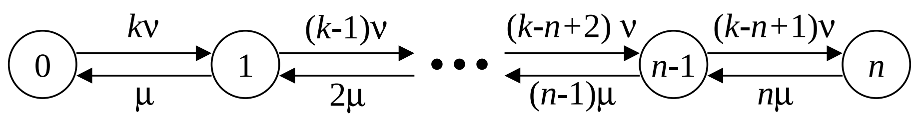
Note: If the system is full when an idle cust. tries to become an active cust., a new idle period starts.

- On-off type customer (note: when active, a customer is in service):



State transition diagram

- Let $X(t)$ denote the number of active customers
  - Assume that $X(t) = i$ at some time t, and consider what happens during a short time interval $(t, t+h]$:
    - if $i < n$, then, with prob. $(k-i)\nu h + o(h)$, an idle customer becomes active (state transition $i \to i+1$)
    - if $i > 0$, then, with prob. $i\mu h + o(h)$, an active customer becomes idle (state transition $i \to i-1$)
- Process $X(t)$ is clearly a Markov process with state transition diagram



- Note that process $X(t)$ is an irreducible birth-death process with a finite state space $S = \{0,1,\ldots,n\}$

Equilibrium distribution (1)

- Local balance equations (LBE):

$$\pi_i (k-i)\nu = \pi_{i+1}(i+1)\mu \qquad \text{(LBE)}$$

$$\Rightarrow \quad \pi_{i+1} = \frac{(k-i)\nu}{(i+1)\mu}\pi_i$$

$$\Rightarrow \quad \pi_i = \frac{k!}{i!(k-i)!}\left(\frac{\nu}{\mu}\right)^i \pi_0 = \binom{k}{i}\left(\frac{\nu}{\mu}\right)^i \pi_0, \quad i = 0,1,\ldots,n$$

- Normalizing condition (N):

$$\sum_{i=0}^{n} \pi_i = \pi_0 \sum_{i=0}^{n} \binom{k}{i}\left(\frac{\nu}{\mu}\right)^i = 1 \qquad \text{(N)}$$

$$\Rightarrow \quad \pi_0 = \left(\sum_{i=0}^{n} \binom{k}{i}\left(\frac{\nu}{\mu}\right)^i \right)^{-1}$$

Equilibrium distribution (2)

- Thus, the equilibrium distribution is a **truncated binomial distribution**:

$$P\{X = i\} = \pi_i = \frac{\binom{k}{i}\left(\frac{\nu}{\mu}\right)^i}{\sum_{j=0}^{n} \binom{k}{j}\left(\frac{\nu}{\mu}\right)^j} = \frac{\binom{k}{i}\left(\frac{\nu}{\nu+\mu}\right)^i \left(\frac{\mu}{\nu+\mu}\right)^{k-i}}{\sum_{j=0}^{n} \binom{k}{j}\left(\frac{\nu}{\nu+\mu}\right)^j \left(\frac{\mu}{\nu+\mu}\right)^{k-j}}, i = 0,1,\ldots,n$$

- Remark (insensitivity):
  - The result is **insensitive both** to the service **and** the idle time distribution, that is: it is valid for **any** service time distribution with mean $1/\mu$ and **any** idlee time distribution with mean $1/\mu$
  - So, instead of the $M/M/n/n/k$ model, we can consider, as well, the more general $G/G/n/n/k$ model

Time blocking

- **Time blocking** B_t = probability that all n channels are occupied at an arbitrary time = the fraction of time that all n channels are occupied
- For a stationary Markov process, this equals the probability π_n of the equilibrium distribution π. Thus,

$$B_t := P\{X = n\} = \pi_n = \frac{\binom{k}{n}\left(\frac{\nu}{\mu}\right)^n}{\sum_{j=0}^{n} \binom{k}{j}\left(\frac{\nu}{\mu}\right)^j}$$

Call blocking (1)

- **Call blocking** B_c = probability that an arriving call finds all n channels occupied = the fraction of calls that are lost
- In the Engset model, however, the “arrivals” do **not** follow a Poisson process. Thus, we cannot utilize the PASTA property any more.
- In fact, the distribution of the state that an “arriving” customer sees differs from the equilibrium distribution. Thus, call blocking B_c does **not** equal time blocking B_t in the Engset model.

Call blocking (2)

- Let π_i* denote the probability that there are i active customers when an idle customer becomes active (which is called an "arrival")
- Consider a long time interval $(0,T)$:
  - During this interval, the average time spent in state i is $\pi_i T$
  - During this time, the average number of "arriving" customers (who all see the system to be in state i) is $(k\text{-}i)\nu\cdot\pi_i T$
  - During the whole interval, the average number of "arriving" customers is $\Sigma_j\ (k\text{-}j)\nu\cdot\pi_j T$
- Thus,

$$\pi_i{}^* = \frac{(k-i)\nu\cdot\pi_i T}{\sum_{j=0}^{n}(k-j)\nu\cdot\pi_j T} = \frac{(k-i)\nu\cdot\pi_i}{\sum_{j=0}^{n}(k-j)\nu\cdot\pi_j},\quad i=0,1,\ldots,n$$

Call blocking (3)

- It can be shown (exercise!) that

$$\pi_i{}^* = \frac{\binom{k-1}{i}\left(\frac{\nu}{\mu}\right)^i}{\sum_{j=0}^{n}\binom{k-1}{j}\left(\frac{\nu}{\mu}\right)^j},\quad i=0,1,\ldots,n$$

- If we write explicitly the dependence of these probabilities on the total number of customers, we get the following result:

$$\pi_i{}^*(k) = \pi_i(k-1),\quad i=0,1,\ldots,n$$

- In other words, an "arriving" customer sees such a system where there is one customer less (she herself!) in equilibrium

Call blocking (4)

- By choosing $i = n$, we get the following formula for the call blocking probability:

$$B_c(k) = \pi_n{}^*(k) = \pi_n(k-1) = B_t(k-1)$$

- Thus, for the Engset model, the call blocking in a system with k customers equals the time blocking in a system with k-1 customers:

$$B_c(k) = B_t(k-1) = \frac{\binom{k-1}{n}\left(\frac{\nu}{\mu}\right)^n}{\sum_{j=0}^{n}\binom{k-1}{j}\left(\frac{\nu}{\mu}\right)^j}$$

- This is **Engset's blocking formula**

THE END