EXERCISES S-38.215: WEEK 1

Exercise 1

Consider a person handling calls in a call center. The interarrival times of calls are exponentially distributed with mean $1/\lambda$. The call holding times are exponentially distributed with mean $1/\mu$. Calls arriving while the person is busy are either put on a holding line (there is only one holding line available) or handled somewhere else by less experienced people (if the holding line is already occupied). Let $p_i(t)$ be the probability that at time t there are i calls present in the system (i.e, either handled by the person or on the holding line), $i = 0, 1, 2$.

(i) Derive the following set of differential equations for $p_i(t), i = 0, 1, 2$,

$$
\begin{aligned}
p_0'(t) &= -\lambda p_0(t) + \mu p_1(t), \\
p_1'(t) &= \lambda p_0(t) - (\lambda + \mu) p_1(t) + \mu p_2(t), \\
p_2'(t) &= \lambda p_1(t) - \mu p_2(t).
\end{aligned}
$$

From now on assume $\lambda = \mu = 1$.

(ii) Solve the differential equations in (i) under the initial condition that at time 0 the system is empty.

(iii) Determine the limiting probabilities $p_i = \lim_{t\to\infty} p_i(t), i = 0, 1, 2$.

(iv) What is in the long run the occupation rate of the person and the fraction of calls that is handled somewhere else?

Exercise 2

Consider a renewal process in which the interoccurrence times between renewals have an Erlang-r distribution. Verify that the residual life time distribution has a probability density function of the form $\sum_{j=1}^{r} p_j f_j(x)$ where $p_j = 1/r$ for all j and $f_j(x)$ is the probability density function of an Erlang-j distribution. Could you give a heuristic explanation of the result?

Exercise 3

Data packets have to be processed in a packet router, which can be modelled as a single server queueing system. Packets arrive according to a Poisson stream with an average rate of 1 packets per 10 ms. For a quarter of the packets the processing time is exponentially distributed with a mean of 10 ms and for the other packets the processing time is exponentially distributed with a mean of 5 ms.

(i) Show that the Laplace-Stieltjes transform of the processing time in ms of an arbitrary packet is given by

$$\widetilde{S}(s) = \frac{1}{4} \cdot \frac{4 + 35s}{(1 + 10s)(1 + 5s)} \,.$$

(ii) Show that the Laplace-Stieltjes transform of the sojourn time (waiting time plus processing time) of an arbitrary packet is given by

$$\widetilde{T}(s) = \frac{5}{32} \cdot \frac{3}{3 + 20s} + \frac{27}{32} \cdot \frac{1}{1 + 20s} \,.$$

(iii) Determine the fraction of packets for which the sojourn time is longer than 30 ms.

(iv) Determine the mean sojourn time.

Exercise 4

A machine produces products in two phases. The first phase is standard and the same for all products. The second phase is customer specific (the finishing touch). The first (resp. second) phase takes an exponential time with a mean of 10 (resp. 2) minutes. Orders for the production of one product arrive according to a Poisson stream with a rate of 3 orders per hour. Orders are processed in order of arrival. Determine the mean production lead time of an order.