HELSINKI UNIVERSITY OF TECHNOLOGY | **Exercise 2**
Networking Laboratory | 16.2.2004
S-38.215 Special Course in Networking Technology, Spring 2004 | Aalto

1. (Lawler, Exercise 1.9) Consider the Markov chain with state space $\{1, 2, 3, 4, 5\}$ and transition matrix

$$P = \begin{bmatrix} 0 & 1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 0 & 1/4 & 3/4 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

   (a) Is the chain irreducible?

   (b) What is the period of the chain?

   (c) What are $p_{1000}(2,1)$, $p_{1000}(2,2)$, $p_{1000}(2,4)$ (approximately)?

   (d) Let T be the first return time to state 1, starting at state 1. What is the distribution of T and what is $E[T]$? What does this say, without any further calculation, about $\pi(1)$?

   (e) Find the invariant distribution $\bar{\pi}$. Use this to find the expected return time to state 2, starting in state 2.

2. Consider the Markov chain with state space $\{0, 1, 2\}$ and transition matrix

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1-p & 0 & p \\ 0 & 1 & 0 \end{bmatrix},$$

   where $0 < p < 1$. Let T_i denote the time of the first visit after time 0 to state i, i.e.

$$T_i = \min\{n \geq 1 | X_n = i\}.$$

   Find the conditional means $E[T_j | X_0 = i]$ for all $i, j \in \{0, 1, 2\}$.

3. (Lawler, Exercise 2.2) Consider the following Markov chain with state space $S = \{0, 1, \ldots\}$. A sequence of positive numbers $p_1, p_2, \ldots$ is given with $\sum_{i=1}^{\infty} p_i = 1$. Whenever the chain reaches state 0, it chooses a new state according to the p_i. Whenever the chain is at a state other than 0, it proceeds deterministically, one step at a time, toward 0. In other words, the chain has transition probability

$$p(x, x-1) = 1, \quad x > 0$$
$$p(0, x) = p_x, \quad x > 0.$$

   This is a recurrent chain since the chain keeps returning to 0. Under what conditions on the p_x is the chain positive recurrent? In this case, what is the limiting probability distribution π? (*Hint*: it may be easier to compute $E[T]$ directly where T is the time of first return to 0 starting at 0.)