Lawler (1995)
Chapter 0
Preliminaries

0.1 Introduction

- Stochastic processes
  - Definition:
    - a collection of random variables X_t indexed by time t
  - Index
    - Discrete time: $\{0,1,2,\ldots\}$
    - Continuous time: subset of $[0,\infty)$
  - State space
    - Discrete: a finite or countably infinite set
    - Continuous: real numbers R or d-dimensional space R^d
- Differential equations (deterministic processes)

$$y'(t) = F(t, y(t))$$

- Markov processes (stochastic processes)
  - The change at time t is determined by the value of the process at time t and not by the values at times before t

0.2 Linear Differential Equations

- Homogeneous differential equation of order n:

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_1 y'(t) + a_0 y(t) = 0 \qquad (0.1)$$

  - General solution:

$$y(t) = c_1 y_1(t) + \cdots + c_n y_n(t)$$

    - Constants $c_1,\ldots,c_n$ determined from the initial conditions:

$$y(0) = b_0,\ y'(0) = b_1,\ \ldots,\ y^{(n-1)}(0) = b_{n-1}$$

    - Functions $y_1(t),\ldots,y_n(t)$ determined as follows:

$$1.\ \text{Try: } y(t) = e^{\lambda t} \quad \Rightarrow \quad y^{(n)}(t) = \lambda^n e^{\lambda t} = \lambda^n y(t)$$

$$2.\ \text{Solve (0.1): } \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$$

$$3.\ \text{Solutions for a root of multiplicity } j:\ e^{\lambda t}, te^{\lambda t}, \ldots, t^{j-1}e^{\lambda t}$$

0.2 Linear Differential Equations

- First-order linear system of differential equations:

$$\bar{y}'(t) = A\bar{y}(t)$$

    - where A is a nxn-matrix and $y(t)$ is a n-vector
  - For any initial n-vector v, there is a unique solution:

$$\bar{y}(t) = e^{tA}\bar{v}$$

    - where

$$e^{tA} = \sum_{j=0}^{\infty} \frac{(tA)^j}{j!}$$

0.3 Linear Difference Equations

- Second-order homogeneous linear difference equation:

$$f(n) = af(n-1) + bf(n+1), \quad K < n < N \qquad (0.2)$$

  - General solution:

$$f(n) = c_1 f_1(n) + c_2 f_2(n)$$

    - Constants c_1 and c_2 determined from the initial conditions:

$$f(n_1) = b_1, \; f(n_2) = b_2$$

    - Functions $f_1(n)$ and $f_2(n)$ determined as follows:

1. Try: $f(n) = \alpha^n$
2. Solve (0.2): $\alpha^n = a\alpha^{n-1} + a\alpha^{n+1} \Rightarrow \alpha = a + b\alpha^2$
3. Solution for a root of multiplicity 1: α^n
4. Solutions for a root of multiplicity 2: $\alpha^n, n\alpha^n$

0.3 Linear Difference Equations

- Special cases of equation (0.2):
  - Recursive formula (when $f(K)$ and $f(K+1)$ are known):

$$f(n+1) = \frac{1}{b}[f(n) - af(n-1)] \qquad (0.3)$$

  - Example: $a + b = 1$
    - Case $a \neq b$ (0.5)
    - Case $a = b$ (0.6)

0.3 Linear Difference Equations

- Homogeneous linear difference equation of order k:

$$f(n+k) = a_0 f(n) + a_1 f(n+1) + \cdots + a_{k-1} f(n+k-1) \quad (0.7)$$

  - General solution:

$$f(n) = c_1 f_1(n) + \cdots + c_k f_k(n)$$

    - Constants $c_1, \ldots, c_k$ determined from the initial conditions:

$$f(n_1) = b_1,\ f(n_2) = b_2, \ldots, f(n_k) = b_k$$

    - Functions $f_1(n), \ldots, f_k(n)$ determined as follows:

$$1.\ \text{Try}: \ f(n) = \alpha^n$$

$$2.\ \text{Solve}\ (0.7): \ \alpha^k = a_0 + a_1\alpha + \cdots + a_{k-1}\alpha^{k-1}$$

$$3.\ \text{Solutions for a root of multiplicity}\ j: \ \alpha^n, n\alpha^n, \ldots, n^{j-1}\alpha^n$$

The End of Chapter 0