Lawler (1995)
Chapter 2
Countable Markov Chains

2.1 Introduction

- Consider a time-homogeneous Markov chain X_n, $n = 0,1,2,\ldots$, with a **countably infinite** state space S
  - Examples: $S = \{0,1,\ldots\}$, $S = Z = \{\ldots,-1,0,1,\ldots\}$, $S = Z^2$
- *Notes*:
  - Markov-property the same as before:

$$P\{X_n = x_n \mid X_0 = x_0,\ldots,X_{n-1} = x_{n-1}\}$$
$$= P\{X_n = x_n \mid X_{n-1} = x_{n-1}\} = p(x_{n-1}, x_n)$$

  - Transition matrix $P = (p(x,y);\ x,y \in S)$ is an infinite matrix.
  - Given an initial distribution $\phi(x) = \mathrm{P}\{X_0 = x\}$, we have

$$P\{X_0 = x_0,\ldots,X_n = x_n\} = \phi(x_0)\,p(x_0,x_1)\cdots p(x_{n-1},x_n)$$

  - Chapman-Kolmogorov equation for the n-step probabilities proved as before

$$p_{m+n}(x,y) = \sum_{z\in S} p_m(x,z)\,p_n(z,y)$$

  - Communication classes (and, thus, irreducibility) defined as before

2.2 Recurrence and Transience

- Consider an irreducible Markov chain X_n with countably infinite state space S
- *Definition*:
  - Markov chain X_n is a **recurrent chain** if, for each state x,

$$P\{X_n = x \text{ for infinitely many } n\} = 1$$

  - Otherwise Markov chain X_n is called a **transient chain**.
- *Notes*:
  - Recurrence and transience are still properties for whole classes
  - Every state of a transient Markov chain is, in fact, visited only a finite number of times (with probability 1)
  - With a finite state space, an irreducible Markov chain is always recurrent, but with a countably infinite state space, it may be transient or recurrent

2.2 Recurrence and Transience

- Consider a Markov chain X_n with countably infinite state space S
  - Fix state x and assume that $X_0 = x$.
  - Let T denote the time for the first return to state x:

$$T = \min\{n \geq 1 \mid X_n = x\}$$

  - Furthermore, let R denote the total number of visits to state x:

$$R = \sum_{n=0}^{\infty} I\{X_n = x\}$$

- *Notes*:
  - The following are the same events:

$$\{X_n = x \text{ for infinitely many } n\} = \{R = \infty\}$$

  - The expectation of R is as follows:

$$E[R] = \sum_{n=0}^{\infty} P\{X_n = x\} = \sum_{n=0}^{\infty} p_n(x,x)$$

2.2 Recurrence and Transience

- *Alternative definition*:
  - State x is **recurrent** if

$$P\{T < \infty\} = 1$$

  - Otherwise, state x is **transient**, i.e.

$$P\{T = \infty\} > 0$$

- *Proposition*:

$$P\{R = k\} = P\{T < \infty\}^{k-1} P\{T = \infty\}$$

- *Proposition*:

$$P\{T < \infty\} = 1 \quad \Leftrightarrow \quad \mathrm{P}\{\mathrm{R} = \infty\} = 1 \quad \Leftrightarrow \quad \mathrm{E}[\mathrm{R}] = \infty$$

- *Proposition*:

$$P\{T = \infty\} > 0 \quad \Leftrightarrow \quad \mathrm{P}\{\mathrm{R} < \infty\} = 1 \quad \Leftrightarrow \quad \mathrm{E}[\mathrm{R}] < \infty$$

2.2 Recurrence and Transience

- *Proposition*:
  - The states of a communication class are either all recurrent or all transient
- *Proposition*:
  - Consider a communication class C.
  - If for some $x \in C$

$$\sum_{y \in C} p(x, y) < 1$$

then class C is transient.

- *Note*:
  - In the case of an infinite state space, this is only a sufficient but not necessary condition
  - Equivalent claim: if class C is recurrent, then for all $x \in C$

$$\sum_{y \in C} p(x, y) = 1$$

2.2 Recurrence and Transience

- *Proposition*:
  - Consider an irreducible Markov chain. Fix state z, and, for each state x, let

$$\alpha(x) = P\{X_n = z \text{ for some } n \geq 0 \mid X_0 = x\}$$

  - The chain is transient if and only if $\alpha(x)$ satifies the following:

$$0 \leq \alpha(x) \leq 1 \tag{2.1}$$

$$\alpha(z) = 1, \quad \inf\{\alpha(x) \mid x \in S\} = 0 \tag{2.2}$$

$$\alpha(x) = \sum_{y \in S} p(x,y)\alpha(y), \quad x \neq z \tag{2.3}$$

- *Notes*:
  - Equations (2.1) and (2.3) together with the first part of (2.2) are clear.
  - So the beef is in the second part of (2.2)

2.3 Positive Recurrence and Null Recurrence

- Consider an irreducible, aperiodic Markov chain X_n with a countably infinite state space S
  - If the state space were finite, the chain would be recurrent with a unique invariant/limiting/stationary distribution
  - However, in the case of a countably infinite state space, the chain may be
    - positive recurrent,
    - null recurrent, or
    - transient
  - Only in the first case the chain has an invariant distibution (being the unique limiting distribution at the same time and leading to a stationary system when used as the initial distribution)
  - In the latter two cases, no invariant/limiting/stationary distribution exists

2.3 Positive Recurrence and Null Recurrence

- *Proposition*:
  - If an irreducible Markov chain is transient, then, for all x,y,

$$\lim_{n\to\infty} p_n(x,y) = 0$$

- *Definition*:
  - A recurrent state x is **null recurrent** if

$$\lim_{n\to\infty} p_n(x,x) = 0$$

  - Otherwise it is called **positive recurrent**.
- *Proposition*:
  - The states of a recurrent communication class are either all null recurrent or all positive recurrent.
- *Proposition*:
  - If an irreducible Markov chain is null recurrent, then, for all x,y,

$$\lim_{n\to\infty} p_n(x,y) = 0$$

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2.3 Positive Recurrence and Null Recurrence

- *Theorem*:
  - Consider an irreducible, aperiodic and positive recurrent Markov chain.
  - It has a unique **limiting** distribution such that,for all x,y,

$$\lim_{n\to\infty} p_n(x,y) = \pi(y) > 0$$

  - The limiting distribution π is the unique **invariant** distribution for the chain, i.e. it satisfies the global balance equations (GBE) together with the normalizing condition (N):

$$\pi(y) = \sum_{x\in S} \pi(x) p(x,y), \quad y \in S \tag{GBE}$$

$$\sum_{x\in S} \pi(x) = 1 \tag{N}$$

  - Starting with π makes the chain **stationary**.
- *Note*:
  - If the chain is irreducible and positive recurrent but periodic, there is still a unique invariant/stationary distribution, but no limiting distribution

2.3 Positive Recurrence and Null Recurrence

- *Proposition*:
  - Let X_n be an irreducible, aperiodic Markov chain. Assume that $X_0 = x$ and let T denote the time for the first return to state x.
  - If X_n is positively recurrent, then

$$E[T] = \frac{1}{\pi(i)} < \infty$$

  - If X_n is null recurrent or transient, then

$$E[T] = \infty$$

- *Idea of the proof*:
  - By a renewal argument applying Blackwell's Theorem. Consecutive visits to state i constitute a renewal process in discrete time.
- *Note*:
  - One way to determine whether or not a chain is positively recurrent is to try to find an invariant distribution, i.e. solve global balance equations (GBE) and take into account the normalizing condition (N)

The End of Chapter 2