Lawler (1995)
Chapter 3
Continuous-Time Markov Chains

3.1 Poisson Process

- Consider a **continuous-time** stochastic process X_t, $t = [0,\infty)$, with state space $\{0,1,2,\dots\}$ and initial state $X_0 = 0$.
- *Definition*:
  - Process X_t is **Poisson process** with rate $\lambda > 0$ if

$$P\{X_{t+h} = k \mid X_t = k\} = 1 - \lambda h + o(h) \qquad (3.1)$$
$$P\{X_{t+h} = k+1 \mid X_t = k\} = \lambda h + o(h) \qquad (3.2)$$

  - As a consequence of (3.1) and (3.2) we have

$$P\{X_{t+h} \neq k, k+1 \mid X_t = k\} = o(h) \qquad (3.3)$$

- Let $P_k(t) = P\{X_t = k\}$.
  - It follows from (3.1)-(3.3) that

$$P_0'(t) = -\lambda P_0(t), \qquad P_k'(t) = \lambda P_{k-1}(t) - \lambda P_k(t), \quad k \geq 1$$

  - Thus,

$$P_0(t) = e^{-\lambda t}, \qquad P_k(t) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}, \quad k \geq 1$$

3.2 Finite State Space

- Consider a **continuous-time** stochastic process X_t, $t = [0,\infty)$, with a **finite** state space S
- *Definition*:
  - Process X_t is a time-homogeneous **continuous-time Markov chain** if

$$P\{X_{t+h} = y \mid X_t = x\} = \alpha(x,y)h + o(h), \quad y \neq x \tag{3.6}$$

for some rate function $\alpha : S \times S \to [0, \infty)$.

  - As a consequence of (3.6), we have

$$P\{X_{t+h} = x \mid X_t = x\} = 1 - \alpha(x)h + o(h), \quad \alpha(x) = \sum_{y \neq x} \alpha(x,y) \tag{3.5}$$

- Markov-property:

$$P\{X_t = y \mid X_r ; 0 \le r \le s\} = P\{X_t = y \mid X_s\}$$

- Time-homogeneity:

$$P\{X_t = y \mid X_s = x\} = P\{X_{t-s} = y \mid X_0 = x\}$$

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3.2 Finite State Space

- Let $p_x(t) = P\{X_t = x\}$.
  - Now it follows from (3.5) and (3.6) that

$$p_x^{'}(t) = -p_x(t)\alpha(x) + \sum_{y \neq x} p_y(t)\alpha(y,x)$$

  - The same in matrix form:

$$\overline{p}'(t) = \overline{p}(t)A \tag{3.7}$$

  - Matrix A is called the **infinitesimal generator** of the chain,

$$[A](x,y) = \begin{cases} -\alpha(x), & x = y \\ \alpha(x,y), & x \neq y \end{cases}$$

  - Equation (3.7) has a well-known solution (see Section 0.2):

$$\overline{p}(t) = \overline{p}(0)e^{tA} = \overline{p}(0)\sum_{n=0}^{\infty} \frac{t^n A^n}{n!}$$

3.2 Finite State Space

- Let $p_t(x,y) = P\{X_t = y | X_0 = x\}$ and the corresponding transition matrix $P_t = (p_t(x,y);\, x,y \in S)$
  - Then

$$P_t^{'} = P_t A, \qquad P_0 = I \tag{3.8}$$

  - It follows that

$$P_t = e^{tA} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}$$

3.2 Finite State Space

- Suppose $T_1,\ldots,T_n$ are independent, exponentially distributed random variables with rates $b_1,\ldots,b_n$, respectively
  - Then $T = \min\{T_1,\ldots,T_n\}$ is exponentially distributed with rate $b_1+\ldots+b_n$,

$$P\{T \geq t\} = e^{-(b_1+\cdots+b_n)}$$

  - Furthermore

$$P\{T = T_i\} = \frac{b_i}{b_1+\cdots+b_n}$$

3.2 Finite State Space

- Consider a continuous-time Markov chain X_t with rate function $\alpha(x,y)$
  - Suppose $X_0 = x$ and let

$$T = \inf\{t > 0 \mid X_t \neq x\}$$

  - By (3.5) and (3.6), we deduce that the expiration rate of T remains constant:

$$P\{T \in (t, t+h] \mid T > t\} \approx P\{X_{t+h} \neq x \mid X_t = x\} = \alpha(x)h + o(h)$$

  - Thus, T, the time spent in state x, is exponentially distributed with rate $\alpha(x)$
  - Furthermore, again by (3.5) and (3.6), we find that the new state is chosen proportionally to the transition rates $\alpha(x,y)$:

$$P\{X_T = y\} = \frac{\alpha(x,y)}{\alpha(x)}, \quad y \neq x$$

  - Thus, $T(x)$, the time spent in state x, can be interpreted as the minimum of independent, exponentially distributed random variables $T(x,y)$, $y \neq x$, with rates $\alpha(x,y)$, respectively

3.2 Finite State Space

- Consider a continuous-time Markov chain X_t with rate function $\alpha(x,y)$
  - Let T_n, $n = 1,2,\ldots$, denote the sequence of jump times, when the state of the system is changed
  - There is a discrete-time Markov chain J_n **embedded** in the jump times:

$$J_0 = X_0, \qquad J_n = X_{T_n^+}, \quad n = 1,2,\ldots$$

  - The transition probabilities $p(x,y) = P\{J_{n+1} = y \mid J_n = x\}$ of this chain are

$$p(x,y) = \begin{cases} \dfrac{\alpha(x,y)}{\alpha(x)}, & y \neq x \\ 0, & y = x \end{cases}$$

- *Definitions*:
  - Two states x and y of the continuous-time Markov chain X_t **communicate** if they communicate in the corresponding discrete-time Markov chain J_n
  - Continuous-time Markov chain X_t is **irreducible** if the corresponding discrete-time Markov chain J_n is irreducible, i.e. there is only one communication class

3.2 Finite State Space

- Consider a continuous-time Markov chain X_t with infinitesimal generator A
- *Definition*:
  - A probability vector π is called an **invariant distribution** for A if

$$\overline{\pi}A = \overline{0}$$

- *Notes*:
  - The system of linear equations given above for the determination of π are called **Global Balance Equations** (GBE):

$$\pi(y)\alpha(y) = \sum_{x \neq y} \pi(x)\alpha(x,y), \quad y \in S \qquad \text{(GBE)}$$

  - Requiring that π is a probability vector (= distribution) is the so called **Normalizing Condition** (N):

$$\sum_{x \in S} \pi(x) = 1 \qquad \text{(N)}$$

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3.2 Finite State Space

- *Proposition*:
  - Suppose π is a **limiting distribution**, i.e. for some initial distribution ϕ, we have

$$\overline{\pi} = \lim_{t \to \infty} \overline{\phi} P_t$$

  - Then it is also an invariant distribution,

$$\overline{\pi}A = \lim_{t \to \infty} (\overline{\phi} P_t) A = \overline{\phi} \lim_{t \to \infty} P_t A \overset{(3.8)}{=} \overline{\phi} \lim_{t \to \infty} P_t' = \lim_{t \to \infty} \overline{\phi} P_t' == 0$$

3.2 Finite State Space

- *Theorem*:
  - Consider an irreducible continuous-time Markov chain with a finite state space
  - It has a unique **limiting** distribution such that, for all x,y,

$$\lim_{t\to\infty} p_t(x,y)=\pi(y)>0$$

  - The limiting distribution π is the unique **invariant** distribution for the chain, i.e. it satisfies the global balance equations (GBE) together with the normalizing condition (N):

$$\pi(y)\alpha(y)=\sum_{x\neq y}\pi(x)\alpha(x,y),\quad y\in S \qquad \text{(GBE)}$$

$$\sum_{x\in S}\pi(x)=1 \qquad \text{(N)}$$

  - Starting with π makes the chain **stationary**.

3.2 Finite State Space

- Let X_n be an irreducible Markov chain with infinitesimal generator A
  - Assume that $X_0 = x$ and let

$$T=\inf\{t>0 \mid X_t\neq x\}$$

  - Denote the first time after 0 that the Markov chain is in a fixed state z by Y,

$$Y=\inf\{t>0 \mid X_t = z\}$$

  - Denote $b(x)=E[Y \mid X_0=x]$. Since $E[T \mid X_0=x]=1/\alpha(x)$, we have

$$\alpha(x)b(x)=1+\sum_{y\neq x,z}\alpha(x,y)b(y)$$

  - Let

$$\widetilde{A}=([A](x,y);x,y\neq z),\qquad \overline{b}=(b(x);x\neq z)$$

  - Then,

$$\overline{0}=\overline{1}+\widetilde{A}\overline{b}\quad\Rightarrow\quad \overline{b}=(-\widetilde{A})^{-1}\overline{1}$$

3.3 Birth-and-Death Processes

- Consider a continuous-time Markov chain X_t, $t = [0,\infty)$, with transition rates $\alpha(x,y)$ and **countably infinite** state space $\{0,1,2,\ldots\}$
- *Definition*:
  - A continuous-time Markov chain is a **birth-and-death** process if

$$\alpha(x,y) = \begin{cases} \lambda_n, & x = n,\ y = n+1,\ n = 0,1,\ldots \\ \mu_n, & x = n,\ y = n-1,\ n = 1,2,\ldots \\ 0, & \text{otherwise} \end{cases}$$

- Thus,

$$P\{X_{t+h} = n \mid X_t = n\} = 1 - (\mu_n + \lambda_n)h + o(h)$$
$$P\{X_{t+h} = n+1 \mid X_t = n\} = \lambda_n h + o(h)$$
$$P\{X_{t+h} = n-1 \mid X_t = n\} = \mu_n h + o(h)$$

- Let $P_n(t) = P\{X_t = n\}$. It follows that

$$P_n'(t) = \mu_{n+1}P_{n+1}(t) + \lambda_{n-1}P_{n-1}(t) - (\mu_n + \lambda_n)P_n(t) \qquad (3.9)$$

3.3 Birth-and-Death Processes

- Consider a birth-and-death process X_t
  - Let J_n denote the corresponding discrete-time Markov chain J_n embedded in the jump times with transition probabilities $p(0,1) = 1$ and for $n > 0$

$$p(n,n-1) = \frac{\mu_n}{\mu_n + \lambda_n}, \qquad p(n,n+1) = \frac{\lambda_n}{\mu_n + \lambda_n}$$

- *Definition*:
  - Birth-and-death process X_t is **irreducible** if the corresponding discrete-time Markov chain J_n is irreducible
- *Note*:
  - Irreducibility is equivalent to the condition that $\lambda_n > 0$ and $\mu_{n+1} > 0$ for all $n = 0,1,\ldots$.
- *Definition*:
  - An irreducible birth-and-death process X_t is **recurrent** if the corresponding discrete-time Markov chain J_n is recurrent. Otherwise it is called **transient**.

3.3 Birth-and-Death Processes

- *Proposition*:
  - An irreducible birth-and-death process X_t is transient if and only if

$$\sum_{n=1}^{\infty} \frac{\mu_1 \cdots \mu_n}{\lambda_1 \cdots \lambda_n} < \infty \tag{3.11}$$

- *Idea of the proof*:
  - For each state n, let

$$\alpha(n) = P\{X_t = 0 \text{ for some } t \geq 0 \mid X_0 = n\}$$

  - From Section 2.2: The chain is transient if and only if $\alpha(n)$ satifies the following:

$$\begin{aligned} &0 \leq \alpha(n) \leq 1 \\ &\alpha(0) = 1, \quad \lim_{n \to \infty} \alpha(n) = 0 \\ &(\mu_n + \lambda_n)\alpha(n) = \mu_n \alpha(n-1) + \lambda_n \alpha(n+1), \quad n > 0 \end{aligned} \tag{3.10}$$

  - It remains to prove that the conditions are equivalent.

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3.3 Birth-and-Death Processes

- *Definition*:
  - An irreducible, recurrent birth-and-death process X_t is **positive recurrent** if it has an invariant distribution, i.e. there is π that satisfies the global balance equations (GBE) together with the normalizing condition (N):

$$\begin{aligned} &\pi(0)\lambda_0 = \pi(1)\mu_1 \\ &\pi(n)(\mu_n + \lambda_n) = \pi(n-1)\lambda_{n+1} + \pi(n+1)\mu_{n+1}, \quad n > 0 \end{aligned} \tag{GBE}$$

$$\sum_{n=0}^{\infty} \pi(n) = 1 \tag{N}$$

  - Otherwise X_t is called **null recurrent**.
- *Notes*:
  - The invariant distribution is the unique limiting distribution:

$$\lim_{t \to \infty} p_t(x, y) = \pi(y) > 0$$

  - An irreducible, recurrent birth-and-death process X_t may be positive recurrent even if the corresponding discrete-time Markov chain J_n is not positive recurrent, and vice versa.

3.3 Birth-and-Death Processes

- *Proposition*:
  - An irreducible birth-and-death process X_t is positive recurrent if and only if

$$\sum_{n=1}^{\infty} \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} < \infty$$

  - The invariant distribution for a positive recurrent process is

$$\pi(n) = \pi(0) \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n}, \qquad \pi(0) = \left(1 + \sum_{n=1}^{\infty} \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} \right)^{-1}$$

- *Idea of the proof*:
  - Solve the global balance equations.

3.4 General Case

- Consider a continuous-time Markov chain X_t with a **countably infinite** state space S and rate function $\alpha(x,y)$ for which

$$\alpha(x) = \sum_{y \neq x} \alpha(x, y) < \infty$$

  - Then

$$P\{X_{t+h} = y \mid X_t = x\} = \alpha(x, y)h + o_x(h)$$

  - If the rates are not bounded, it is possible for the chain to have an explosion in finite time. However, we assume here that no such explosion happens.

3.4 General Case

- Let $p_t(x,y) = P\{X_t = y | X_0 = x\}$.
  - If

$$\sum_y p_t(x,y) o_y(h) = o(h) \qquad (3.14)$$

then we have the following **forward equations** for the chain:

$$p_t'(x,y) = -p_t(x,y)\alpha(y) + \sum_{z \neq y} p_t(x,z)\alpha(z,y) \qquad \text{(FWD)}$$

  - However, the corresponding **backward equations** need no additional conditions:

$$p_t'(x,y) = -\alpha(x) p_t(x,y) + \sum_{z \neq x} \alpha(x,z) p_t(z,y) \qquad \text{(BWD)}$$

  - In the case of a finite state space with infinitesimal generator A, we have the following matrix equations (cf. (3.8)) with the same solution $P_t = e^{At}$:

$$P_t' = P_t A \qquad \text{(FWD)}$$

$$P_t' = A P_t \qquad \text{(BWD)}$$

The End of Chapter 3