Lawler (1995)
Chapter 5
Martingales

5.1 Conditional Expectation

- *Definition*:
  - Consider discrete random variables X and Y. Let $S_X = \{x \mid P\{X = x\} > 0\}$.
  - **Conditional expectation** of Y given that $X = x$, where $x \in S_X$, is defined by

$$E[Y \mid X](x) = E[Y \mid X = x] = \sum_y yP\{Y = y \mid X = x\}$$

- *Notes*:
  - Conditional expectation $E[Y|X]$ is a random variable having a constant value for all those realizations for which X is constant:

$$E[Y \mid X] = \sum_{x \in S_X} I\{X = x\} E[Y \mid X = x]$$

  - It is clearly a function of X :

$$E[Y \mid X] = \phi(X)$$

  - Moreover, its expectation equals $E[Y]$:

$$E[E[Y \mid X]] = \sum_y yP\{E[Y \mid X] = y\} = \sum_x E[Y \mid X = x] P\{X = x\}$$
$$= \sum_x \sum_y yP\{Y = y \mid X = x\} P\{X = x\} = \sum_y yP\{Y = y\} = E[Y]$$

5.1 Conditional Expectation

- *Definitions*:
  - Consider random variables $X_1,\ldots, X_n$. Denote by F_n the σ-algebra (i.e. collection of events) generated by these random variables,

$$F_n = \sigma(X_1,\ldots,X_n)$$

  - Random variable Y is F_n**-measurable** if it depends only on $X_1,\ldots, X_n$, i.e. there is a function f such that

$$Y = f(X_1,\ldots,X_n)$$

  - Event $A \in F_n$ if the corresponding indicator function $I(A)$ is F_n-measurable
  - Increasing sequence of σ-algebras $F_1, F_2, \ldots$ is called a **history**. Usually the whole history is briefly denoted by F_n.

5.1 Conditional Expectation

- *Definition*:
  - Consider random variables $X_1,\ldots, X_n$ and Y.
  - Random variable $E[Y|X]$ is the **conditional expectation** of Y given F_n if
    - It is F_n-measurable, i.e. there is a function ϕ such that

$$E[Y \mid F_n] = \phi(X_1,\ldots,X_n)$$

    - For all events $A \in F_n$ (depending only on $X_1,\ldots, X_n$),

$$E[YI_A] = E[E[Y \mid F_n]I_A] \tag{5.1}$$

- *Note*:
  - By applying the second condition for the whole sample space Ω, we get

$$E[Y] = E[YI_\Omega] = E[E[Y \mid F_n]I_\Omega] = E[E[Y \mid F_n]]$$

5.1 Conditional Expectation

- *Proposition*:
  - For any $m < n$,

$$E[E[Y \mid F_n] \mid F_m] = E[E[Y \mid F_m] \mid F_n] = E[Y \mid F_m] \tag{5.5}$$

  - For any constants a and b,

$$E[aY_1 + bY_2 \mid F_n] = aE[Y_1 \mid F_n] + bE[Y_2 \mid F_n] \tag{5.3}$$

  - For all random variables Z that are F_n-measurable,

$$E[ZY \mid F_n] = ZE[Y \mid F_n] \tag{5.7}$$

  - If Y is F_n-measurable, then

$$E[Y \mid F_n] = Y \tag{5.4}$$

  - If Y is independent of F_n, then

$$E[Y \mid F_n] = E[Y] \tag{5.6}$$

5.2 Definition and Examples

- *Definition*:
  - Let $X_0, X_1, \ldots$ be a sequence of random variables. Let $F_n = \sigma(X_0, \ldots, X_n)$ denote the σ-algebra generated by random variables $X_0, \ldots, X_n$.
  - Let $M_0, M_1, \ldots$ be another sequence of random variables s.t., for all n,

$$E[|M_n|] < \infty$$

  - Sequence M_n is a **martingale** w.r.t. history F_n if, for all n,

$$E[M_{n+1} \mid F_n] = M_n$$

- *Note*:
  - By applying (5.5), it is easy to see that for all k

$$E[M_{n+k} \mid F_n] = M_n \tag{5.9}$$

  - A Markov chain X_n is a martingale w.r.t. to its own history $\sigma(X_0, \ldots, X_n)$ if and only if, for all n, $E[|X_n|] < \infty$ and

$$E[X_{n+1} \mid X_n] = X_n$$

5.2 Definition and Examples

- *Definition*:
  - Let $X_0, X_1, \ldots$ be a sequence of random variables. Let $F_n = \sigma(X_0, \ldots, X_n)$ denote the σ-algebra generated by random variables $X_0, \ldots, X_n$.
  - Let $M_0, M_1, \ldots$ be another sequence of random variables s.t., for all n,

$$E[|M_n|] < \infty$$

  - Sequence M_n is a **submartingale** w.r.t. history F_n if, for all n,

$$E[M_{n+1} \mid F_n] \geq M_n$$

  - Sequence M_n is a **supermartingale** w.r.t. history F_n if, for all n,

$$E[M_{n+1} \mid F_n] \leq M_n$$

- *Note*:
  - Sequence M_n is a martingale if and only if it is both a submartingale and a supermartingale

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5.3 Optional Sampling Theorem

- *Definition*:
  - Let $X_0, X_1, \ldots$ be a sequence of random variables. Let $F_n = \sigma(X_0, \ldots, X_n)$ denote the σ-algebra generated by random variables $X_0, \ldots, X_n$.
  - Random variable T taking values in $\{0,1,\ldots\} \cup \{\infty\}$ is a **stopping time** w.r.t. the history $F_0, F_1, \ldots$ if for all n

$$\{T = n\} \in F_n$$

- *Proposition*:

$$\{T \leq n\} \in F_n, \qquad \{T > n\} \in F_n$$

- *Proof*:

$$\{T \leq n\} = \bigcup_{m=0}^{n} \{T = m\} \in F_n, \qquad \{T > n\} = \{T \leq n\}^{\mathrm{c}} \in F_n$$

- *Notes*:
  - Each n is a stopping time
  - If T is a stopping time, then $\min\{T,n\}$ is a stopping time
  - If T and U are stopping times, then $\min\{T,U\}$ is a stopping time

5.3 Optional Sampling Theorem

- *Fact*:
  - Suppose that M_n is a martingale and T is a stopping time w.r.t. history F_n
  - If T is bounded (i.e. there is constant K s.t. $P\{T \le K\} = 1$), then

$$E[M_T \mid F_0] = M_0$$

  - In particular, $E[M_T] = E[M_0]$.
- *Optional Sampling Theorem*:
  - Suppose that M_n is a martingale and T is a stopping time w.r.t. history F_n
  - If T is finite (i.e. $P\{T < \infty\} = 1$) and

$$E[|M_T|] < \infty, \tag{5.10}$$

$$\lim_{n\to\infty} E[|M_n| I\{T > n\}] = 0 \tag{5.11}$$

then

$$E[M_T] = E[M_0]$$

- *Note*:
  - Eqs. (5.10) and (5.11) are satisfied if T is finite and M_n is a bounded martingale (i.e. there is constant C s.t. $P\{|M_n| \le C\} = 1$) for all n

5.4 Uniform Integrability

- *Note*:
  - If X is a random variable with $E[|X|] < \infty$, then for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$P(A) < \delta \quad \Rightarrow \quad E[|X| I_A] < \varepsilon$$

- *Definition*:
  - A sequence of random variables $X_0, X_1, \ldots$ is **uniformly integrable** if for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$P(A) < \delta \quad \Rightarrow \quad E[|X_n| I_A] < \varepsilon \quad \text{for all } n \tag{5.12}$$

- *Optional Sampling Theorem*:
  - Suppose that M_n is a uniformly integrable martingale and T is a stopping time w.r.t. history F_n such that $P\{T < \infty\} = 1$ and $E[|M_T|] < \infty$, then

$$E[M_T] = E[M_0]$$

- *Proof*:
  - Eq. (5.11) follows from the uniform integrability, since $P\{T > n\} \to 0$.

5.5 Martingale Convergence Theorem

- *Martingale Convergence Theorem*:
  - Suppose that M_n is such a martingale that there is constant C with $E[| M_n |] \le C$ for all n.
  - Then there exists such a random variable M_∞ that

$$\lim_{n \to \infty} M_n = M_\infty$$

- *Fact*:
  - Suppose that M_n is a uniformly integrable martingale, then there exists such a random variable M_∞ that

$$\lim_{n \to \infty} M_n = M_\infty$$

and

$$E[M_\infty] = E[M_0]$$

The End of Chapter 5