Lawler (1995)
Chapter 6
Renewal Processes

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6.1 Introduction

- *Definition*:
  - Let $T_1, T_2, \ldots$ be independent and identically distributed (i.i.d.) nonnegative random variables with mean $\mu = E[T_i] > 0$. The **renewal process** associated with renewal sequence T_i is the process N_t with

$$N_t = \begin{cases} 0, & t < T_1 \\ \max\{n = 1,2,\ldots \mid T_1 + \ldots + T_n \le t\}, & t \ge T_1 \end{cases}$$

  - Thus, N_t denotes the number of events occurred up to time t.
- *Definition*:
  - Let $T_1, T_2, \ldots$ be independent and identically distributed (i.i.d.) nonnegative random variables with mean $\mu = E[T_i] > 0$. Furthermore, let Y another nonnegative random variable indepedndent of the sequence T_i. The **renewal process** associated with random variable Y and sequence T_i is the process N_t with

$$N_t = \begin{cases} 0, & t < Y \\ \min\{n = 1,2,\ldots \mid Y + T_1 + \ldots + T_n > t\}, & t \ge Y \end{cases} \quad (6.1)$$

6.1 Introduction

- *Definitions*:
  - Let N_t be a renewal process associated with renewal sequence T_i. The **age** associated with renewal process N_t is the process A_t with

$$A_t = \begin{cases} t, & t < T_1 \\ t - (T_1 + \ldots + T_{N_t}), & t \geq T_1 \end{cases}$$

  - The **residual life** associated with renewal process N_t is the process B_t with

$$B_t = T_{N_t+1} - A_t = \inf\{s : N_{t+s} > N_t\}$$

  - The **total lifetime** associated with renewal process N_t is the process C_t with

$$C_t = T_{N_t+1} = A_t + B_t$$

- *Note*:
  - Renewal process N_t alone is not a Markov process (unless the interarrival times T_i be exponential) but the pair (N_t, A_t) constitutes a Markov process

6.1 Introduction

- *Strong Law of Large Numbers (SLLN) for renewal processes*:
  - Let N_t be a renewal process associated with renewal sequence T_i with mean μ.
  - Then, with probability 1,

$$\lim_{t\to\infty} \frac{N_t}{t} = \frac{1}{\mu} \tag{6.2}$$

- *Central Limit Theorem (CLT) for renewal processes*:
  - Let N_t be a renewal process associated with renewal sequence T_i with mean μ and variance σ^2.
  - Then

$$\frac{N_t - \mu^{-1} t}{\sigma \mu^{-3/2} t^{1/2}} \xrightarrow{\text{i.d.}} N(0,1)$$

6.2 Renewal Equation

- Consider a renewal process N_t associated with renewal sequence T_i with mean μ and distribution function F.
- *Definition*:
  - The **renewal function** U is defined by

$$U(t) = 1 + E[N_t]$$

- *Elementary Renewal Theorem*:

$$\lim_{t \to \infty} \frac{U(t)}{t} = \frac{1}{\mu} \tag{6.3}$$

- Note:
  - It follows that

$$\lim_{t \to \infty} \frac{E[N_t]}{t} = \frac{1}{\mu}$$

6.2 Renewal Equation

- *Definition*:
  - Let X be a nonnegative random variable.
  - It has a **lattice distribution** if there is $a > 0$ such that

$$P\{X \in \{0, a, 2a, \ldots\}\} = 1$$

  - In this case a is called the **period** of the distribution.
  - Otherwise it has a **nonlattice distribution**.
- *Blackwell's Theorem*:
  - If T_1, T_2 , … have a nonlattice distribution, then for all $r > 0$

$$\lim_{t \to \infty} U(t+r) - U(t) = \frac{r}{\mu} \tag{6.4}$$

  - If T_1, T_2 , … have a lattice distribution with period a, then

$$\lim_{n \to \infty} U((n+1)a) - U(na) = \frac{a}{\mu}$$

6.2 Renewal Equation

- *Definition*:
  - Let F and G be distributions of nonnegative random variables.
  - The **convolution** $F*G$ is another distribution defined by

$$[F*G](t) = \int_0^t F(t-s)dG(s) = \int_0^t G(t-s)dF(s)$$

- *Notes*:
  - Let X and Y be independent nonnegative random variables with distributions F and G, respectively. Then the convolution $F*G$ is the distribution of their sum $X+Y$,

$$P\{X+Y \le t\} = [F*G](t)$$

Chapter 6: Renewal Processes

6.2 Renewal Equation

- *Definition*:
  - Let F be a distribution of a nonnegative random variable.
  - The **iterated convolution** $F^{(n)}$ is defined by

$$F^{(0)}(t) = 1 \qquad \text{for all } t \ge 0$$
$$F^{(n+1)} = F^{(n)} * F \qquad \text{for all } n \ge 0$$

- *Proposition*:

$$U = \sum_{n=0}^{\infty} F^{(n)}$$

- *Proof*:

$$\begin{aligned} U(t) &= 1 + E[N_t] \\ &= 1 + \sum_{n=1}^{\infty} P\{N_t \ge n\} \\ &= 1 + \sum_{n=1}^{\infty} P\{T_1 + \ldots + T_n \le t\} = \sum_{n=0}^{\infty} F^{(n)}(t) \end{aligned}$$

6.2 Renewal Equation

- Consider a renewal process N_t associated with renewal sequence T_i with distribution function F and renewal function U.
- *Definition*:
  - Let h be a nonnegative function defined on $[0,\infty)$. Function ϕ defined on $[0,\infty)$ satisfies the **renewal equation** if

$$\phi = h + \phi * F$$

  - Equivalently, for all t,

$$\phi(t) = h(t) + \int_0^t \phi(t-s)dF(s) \qquad (6.6)$$

- *Proposition*:
  - There is a unique solution to the renewal equation,

$$\phi = h * U$$

- *Note*:
  - If F is nonlattice, h is bounded and $\int_0^\infty |h(t)|dt < \infty$, then

$$\phi(\infty) = \lim_{t\to\infty} \phi(t) = \lim_{t\to\infty} \int_0^t h(t-s)dU(s) = \frac{1}{\mu}\int_0^\infty h(s)ds \qquad (6.8)$$

6.2 Renewal Equation

- Consider then the age process A_t of the renewal process N_t.
- Denote its steady state distribution by

$$\Psi_A(x) = \lim_{t\to\infty} P\{A_t \le x\}$$

- *Proposition*:
  - Probabilities $P\{A_t \le x\}$ satisfy the following renewal equation:

$$P\{A_t \le x\} = 1_{[0,x]}(t)(1-F(t)) + \int_0^t P\{A_{t-s} \le x\}dF(s) \qquad (6.5)$$

- *Proposition*:
  - Steady state distribution ($t \to \infty$) is as follows:

$$\Psi_A(x) = \frac{1}{\mu}\int_0^x (1-F(y))dy$$

  - It is a continuous distribution with density

$$\psi_A(x) = \frac{1}{\mu}(1-F(x))$$

6.2 Renewal Equation

- Consider then the residual life process B_t of the renewal process N_t.
- Denote its steady state distribution by

$$\Psi_B(x) = \lim_{t\to\infty} P\{B_t \le x\}$$

- *Proposition*:
  - Probabilities $P\{B_t \le x\}$ satisfy the following renewal equation:

$$P\{B_t \le x\} = (F(t+x) - F(t)) + \int_0^t P\{B_{t-s} \le x\} dF(s)$$

- *Proposition*:
  - Steady state distribution ($t \to \infty$) is the same as for the age process:

$$\Psi_B(x) = \frac{1}{\mu}\int_0^x (1 - F(y))dy$$

  - It is a continuous distribution with density

$$\psi_B(x) = \frac{1}{\mu}(1 - F(x))$$

- *Note*:
  - Starting with Y that has distribution Ψ_B results in stationary increments

6.2 Renewal Equation

- Consider finally the total lifetime process C_t of the renewal process N_t.
- Denote its steady state distribution by

$$\Psi_C(x) = \lim_{t\to\infty} P\{C_t \le x\}$$

- *Proposition*:
  - Probabilities $P\{C_t \le x\}$ satisfy the following renewal equation:

$$P\{C_t \le x\} = 1_{[0,x]}(t)(F(x) - F(t)) + \int_0^t P\{C_{t-s} \le x\} dF(s)$$

- *Proposition*:
  - Steady state distribution ($t \to \infty$) is as follows:

$$\Psi_C(x) = \frac{1}{\mu}\int_0^x (F(x) - F(y))dy$$

  - If F is a continuous distribution with density f, then the steady state distribution of the total lifetime is a continuous distribution with density

$$\psi_C(x) = \frac{1}{\mu} x f(x)$$

6.3 Discrete Renewal Processes

- Consider a discrete-time renewal process N_j associated with renewal sequence T_i with mean μ and a *lattice* distribution function F with period $a = 1$ and $F(0) = 0$.
- Thus, we have

$$N_j = \begin{cases} 0, & j < T_1 \\ \max\{n = 1,2,\ldots \mid T_1 + \ldots + T_n \le j\}, & j \ge T_1 \end{cases}$$

- Denote the point probabilities by

$$p_n = P\{T_i = n\} = F(n) - F(n-1)$$

- *Proposition*:
  - Age process A_j is a discrete-time Markov chain with transition probabilities

$$p(n,0) = \lambda_{n+1}, \qquad p(n,n+1) = 1 - \lambda_{n+1}$$

where

$$\lambda_n = \frac{p_n}{1-F(n-1)}, \qquad 1 - \lambda_n = \frac{1-F(n)}{1-F(n-1)}$$

6.3 Discrete Renewal Processes

- *Proposition*:
  - Age process A_j is irreducible, aperiodic, and positive recurrent with steady state probabilities

$$\psi_A(0) = \frac{1}{\mu}, \qquad \psi_A(n) = \frac{1}{\mu}(1 - F(n)), \qquad n = 1,2,\ldots$$

- *Blackwell's Theorem*:

$$\lim_{j\to\infty} U(j+1) - U(j) = \lim_{j\to\infty} P\{A_{j+1} = 0\} = \psi_A(0) = \frac{1}{\mu}$$

- *Proposition*:
  - Residual life process B_j has steady state distribution

$$\psi_B(n) = \psi_A(n-1) = \frac{1}{\mu}(1 - F(n-1)), \qquad n = 1,2,\ldots$$

- *Proposition*:
  - Total life time process C_j has steady state distribution

$$\psi_C(n) = \frac{1}{\mu} n p_n, \qquad n = 1,2,\ldots$$

6.4 M/G/1 and G/M/1 queues

- M/G/1 queue
  - Renewal every time the queue becomes nonempty
    - Cycle = busy time + idle time
    - Busy time and idle time independent
    - Idle time exponentially distributed
  - Renewal every time the queue becomes empty
    - Cycle = idle time + busy time
    - Busy time and idle time independent
    - Idle time exponentially distributed

- G/M/1 queue
  - Renewal every time the queue becomes nonempty
    - Cycle = busy time + idle time
    - Busy time and idle time dependent
  - No renewal when the queue becomes empty

The End of Chapter 6