Lawler (1995)
Chapter 8
Brownian Motion

8.1 Introduction

- *Limit of a random walk*:
  - Consider a symmetric random walk $S_n = X_1 + \ldots + X_n$ where

$$P\{X_i = 1\} = P\{X_i = -1\} = 1/2, \qquad E[X_i] = 0, \qquad D^2[X_i] = 1$$

  - Random walk S_n is a martingale
  - Define then process $W_t^{(N)}$ to be the piecewisely linear process with

$$W_{k/N}^{(N)} = \tfrac{1}{\sqrt{N}} S_k$$

  - Its mean and variance are

$$E[W_{k/N}^{(N)}] = \tfrac{1}{\sqrt{N}} E[S_k] = 0, \qquad D^2[W_{k/N}^{(N)}] = \tfrac{1}{N} D^2[S_k] = \tfrac{k}{N}$$

  - In particular,

$$E[W_1^{(N)}] = 0, \qquad D^2[W_1^{(N)}] = 1$$

  - By Central Limit Theorem,

$$W_t^{(N)} \underset{N\to\infty}{\overset{\text{i.d.}}{\to}} N(0,t)$$

8.1 Introduction

- *Definition*:
  - A **Brownian motion** or a **Wiener process** with variance parameter σ^2 is a stochastic process X_t taking values in the real numbers satisfying
    - (i) $X_0 = 0$;
    - (ii) For any $s_1 \le t_1 \le s_2 \le t_2 \le \ldots \le s_n \le t_n$, the random variables $X_{t_1} - X_{s_1}, X_{t_2} - X_{s_2}, \ldots, X_{t_n} - X_{s_n}$ are independent;
    - (iii) For any $s < t$, the random variable $X_t - X_s$ has a normal distribution with mean 0 and variance $\sigma^2(t - s)$;
    - (iv) The paths are continuous, i.e., X_t is a continuous function of t.
  - **Standard Brownian motion** is a Brownian motion with $\sigma^2 = 1$.
  - **Brownian motion starting at** x is defined by $Y_t = x + X_t$.
- *Notes*:
  - Process $Z_t = X_t/\sigma$ is a standard Brownian motion
  - Process $Y_t = a^{-1/2} X_{at}$ is a Brownian motion with variance parameter σ^2
- *Fact*:
  - The path of a Brownian motion X_t is nowhere differentiable

8.2 Markov Property

- Consider a Brownian motion X_t.
  - Let $F_t = \sigma(X_s;\ s \le t)$ denote the sigma-algebra generated by the process up to time t.
  - Due to independent increments, X_t is a Markov process (in a continuous state space) with the following **Markov property**:

$$E[X_{t+h} \mid F_t] = E[X_{t+h} \mid X_t]$$

  - Furthermore, since increments have zero mean (i.e., no drift), we have

$$E[X_{t+h} \mid F_t] = E[X_{t+h} \mid X_t] = X_t + E[X_{t+h} - X_t] = X_t$$

  - Thus, X_t is a martingale.
  - Since increments have a normal distribution, the transition density $p_t(x,y) = P\{X_t \in dy \mid X_0 = x\}$ is as follows:

$$p_t(x,y) = \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(y-x)^2}{2\sigma^2 t}}$$

8.2 Markov Property

- *Chapman-Kolmogorov equation*:

$$p_{s+t}(x,y)=\int_{-\infty}^{\infty}p_s(x,z)p_t(z,y)dz$$

- *Markov property*:
  - Process $Y_t = X_{s+t} - X_s$ is a Brownian motion independent of F_s
- *Strong Markov property*:
  - Let T be a stopping time and $F_T = \sigma(X_s;\, s \le T)$ denote the sigma-algebra generated by the process up to time T.
  - Process $Y_t = X_{T+t} - X_T$ is a Brownian motion independent of F_T
- *Reflection principle*:
  - Consider a Brownian motion X_t starting at a. Let $b > a$. Then, for all $t > 0$,

$$P\{X_s \ge b \ \text{ for some } \ 0 \le s \le t\} = 2P\{X_t \ge b\}$$

- *Example*:
  - Let $t > 1$. Then, by applying the reflection principle, we may deduce that

$$P\{X_s = 0 \ \text{ for some } \ 1 \le s \le t\} = 1 - \frac{2}{\pi}\arctan\frac{1}{\sqrt{t-1}} \to 1 \quad (\text{as } \ t \to \infty)$$

8.3 Zero Set of Brownian Motion

- *Note on scalings*:
  - Let X_t be a standard Brownian motion
  - Process $Y_t = t\,X_{1/t}$ is a standard Brownian motion
- Consider a standard Brownian motion X_t.
  - Let

$$Z = \{t : X_t = 0\}$$

  - According to the example presented in the previous slide,

$$P\{Z \cap [1,\infty) \ne \phi\} = 1$$

  - Thus, with probability 1, process X_t returns to the origin, i.e., X_t is **recurrent** taking both positive and negative values for arbitrarily **large** values of t.
  - Now $Y_t = t\,X_{1/t}$ is also a standard Brownian motion taking both positive and negative values for arbitrarily **small** values of t.
  - Thus,

$$P\{Z \cap (0,\delta) \ne \phi\} = 1 \qquad \text{for all } \delta > 0$$

8.7 Brownian Motion with Drift

- *Definition*:
  - Process Y_t is a **Brownian motion with drift** μ if $Y_t = X_t + \mu t$, where X_t is a Brownian motion
- *Properties*:
  - (i) $Y_0 = 0$;
  - (ii) For any $s_1 \le t_1 \le s_2 \le t_2 \le \ldots \le s_n \le t_n$, the random variables $Y_{t_1} - Y_{s_1}, Y_{t_2} - Y_{s_2}, \ldots, Y_{t_n} - Y_{s_n}$ are independent;
  - (iii) For any $s < t$, the random variable $Y_t - Y_s$ has a normal distribution with mean $\mu(t - s)$ and variance $\sigma^2(t - s)$;
  - (iv) The paths are continuous, i.e., Y_t is a continuous function of t.
- *Transition density*:

$$p_t(x, y) = \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(y-x-\mu t)^2}{2\sigma^2 t}}$$

- *Chapman-Kolmogorov equation*:

$$p_{s+t}(x, y) = \int_{-\infty}^{\infty} p_s(x, z) p_t(z, y) dz$$

Chapter 8: Brownian Motion

The End of Chapter 8