Lawler (1995)
Chapter 9
Stochastic Integration

9.1 Integration with Respect to Random Walk

- *Definition*:
  - Consider symmetric random walk $S_n = X_1 + \ldots + X_n$ with $S_0 = 0$ and

$$P\{X_i = 1\} = P\{X_i = -1\} = 1/2, \quad E[X_i] = 0, \quad D^2[X_i] = 1$$

  - Let $F_n = \sigma(X_1,\ldots,X_n)$ denote the σ-algebra generated by random variables $X_1,\ldots,X_n$.
  - Let B_n be F_{n-1}-measurable random variable (called **bet**) with $E[B_n{}^2] < \infty$.
  - The **integral** of process B_n with respect to random walk S_n is defined by

$$Z_n = \sum_{i=1}^{n} B_i X_i = \sum_{i=1}^{n} B_i (S_i - S_{i-1}) = \sum_{i=1}^{n} B_i \Delta S_i$$

- *Properties*:
  - Process Z_n is a martingale (as shown in Ch. 5), i.e.

$$E[Z_{n+m} \mid F_n] = Z_n$$

  - It has zero mean, $E[Z_n] = 0$, and a finite variance,

$$D^2[Z_n] = \sum_{i=1}^{n} E[B_i^2] < \infty$$

9.2 Integration with Respect to Brownian Motion

- *Definition*:
  - Consider a standard Brownian motion W_t.
  - Let $F_t = \sigma(W_s; s \leq t)$ denote the sigma-algebra generated by the process up to time t.
  - Let Y_t be F_t-measurable process (called **strategy**) with right-continuous paths having left limits, $E[Y_t^2] < \infty$ and

$$\int_0^t E[Y_s^2]ds < \infty$$

  - Furthermore, we assume that Y_t is a **simple strategy**, i.e. there are $0 = t_0 < t_1 < \ldots < t_n < t_{n+1} = \infty$ such that

$$Y_t = \sum_{i=0}^n Y_i 1_{[t_i, t_{i+1})}(t)$$

  - The **integral** of the simple strategy Y_t with respect to Brownian motion W_t is defined by

$$Z_t = \int_0^t Y_s dW_s = \sum_{i=0}^{j-1} Y_i (W_{t_{i+1}} - W_{t_i}) + Y_j (W_t - W_{t_j}), \quad \text{for all } t_j \leq t < t_{j+1}$$

9.2 Integration with Respect to Brownian Motion

- *Properties*:
  - Integral is **linear**, i.e. for all simple strategies X_t and Y_t and real numbers a and b, the strategy $aX_t + bY_t$ is simple and

$$\int_0^t (aX_s + bY_s)dW_s = a\int_0^t X_s dW_s + b\int_0^t Y_s dW_s$$

  - Integral $Z_t = \int_0^t X_s dW_s$ is a **martingale** with respect to history F_t, i.e. Z_t is F_t-measurable, $E[| Z_t |] < \infty$ and

$$E[Z_{t+s} \mid F_t] = Z_t \tag{9.1}$$

  - In particular,

$$E[Z_t] = E[E[Z_t \mid F_0]] = E[Z_0] = 0$$

  - Moreover, it has a finite variance,

$$D^2[Z_t] = \int_0^t E[Y_s^2]ds < \infty \tag{9.2}$$

9.2 Integration with Respect to Brownian Motion

- *Definition*:
  - Let Y_t be F_t-measurable process (called **strategy**) with right-continuous paths having left limits, $E[Y_t^2] < \infty$ and

$$\int_0^t E[Y_s^2]ds < \infty$$

  - Define for all n, the following simple strategies

$$Y_t^{(n)} = \sum_k n \int_{(k-1)/n}^{k/n} Y_s ds \, 1_{[k/n,(k+1)/n)}(t)$$

  - The **integral** of the strategy Y_t with respect to Brownian motion W_t is defined as the limit

$$Z_t = \int_0^t Y_s dW_s = \lim_{n\to\infty} \int_0^t Y_s^{(n)} dW_s$$

- *Properties*:
  - It has the same properties as the integral of a simple strategy: it is linear and a martingale with

$$E[Z_t] = 0, \qquad D^2[Z_t] = \int_0^t E[Y_s^2]ds < \infty$$

Chapter 9: Stochastic Integration

9.3 Ito's Formula

- Consider a standard Brownian motion W_t.
  - Let f be a function with at least two continuous derivatives. Then

$$f(W_t) - f(W_0) = \sum_{j=0}^{n-1}[f(W_{(j+1)t/n}) - f(W_{jt/n})] \approx$$

$$\sum_{j=0}^{n-1} f'(W_{jt/n})[W_{(j+1)t/n} - W_{jt/n}] + \frac{1}{2}\sum_{j=0}^{n-1} f''(W_{jt/n})[W_{(j+1)t/n} - W_{jt/n}]^2$$

  - Define

$$Q_t(g) = \lim_{n\to\infty} \sum_{j=0}^{n-1} g(W_{jt/n})[W_{(j+1)t/n} - W_{jt/n}]^2$$

  - If g is continuous, then

$$Q_t(g) = \int_0^t g(W_s)ds$$

- *Ito's formula*:
  - If f is a function with two continuous derivatives. Then

$$f(W_t) - f(W_0) = \int_0^t f'(W_s)dW_s + \frac{1}{2}\int_0^t f''(W_s)ds$$

9.4 Simulation

- Let Z_t denote a Brownian motion with drift process X_t and variance process Y_t^2, i.e.

$$Z_t = \int_0^t X_s ds + \int_0^t Y_s dW_s$$

  - In differential form this is written

$$dZ_t = X_t dt + Y_t dW_t$$

- Consider then the following **stochastic differential equation**:

$$dX_t = a(X_t)dt + b(X_t)dW_t$$

  - The solution is a process X_t that at any particular time looks like a Brownian motion with drift process $a(X_t)$ and variance process $b(X_t)$
  - To simulate this process,
    - choose some small number Δt and i.i.d. random variables $Y_1, Y_2, \ldots$ such that $P\{Y_i = 1\} = P\{Y_i = -1\} = 1/2$
    - set $X_0 = 0$ and let

$$X_{n\Delta t} = X_{(n-1)\Delta t} + a(X_{(n-1)\Delta t})\Delta t + b(X_{(n-1)\Delta t})\sqrt{\Delta t}Y_n$$

The End of Chapter 9