1. Birth-death process, $\mu_n = n\mu$, and $\lambda_n = n\lambda + \theta$, models the population size with linear growth and immigration. Let the initial size be x_0, i.e. $X(0) = x_0$, and let $m(t)$ denote the expected size of the population at time t, $m(t) = \mathrm{E}[X(t)]$. Determine $m(t)$.

   Hint: What is the expected value of $X(t+h)$ when conditioned on $X(t)$, $\mathrm{E}[X(t+h)|X(t)]$, when h is small? By taking an expectation from the both sides of the equation (tower property) and by letting h go to zero you obtain a differential equation for $m(t)$.

2. Let $N(t)$, $t \geq 0$, be a Poisson process with rate λ. Let S_n denote the time of occurance of the nth event. Find

   a) $\mathrm{E}[S_4]$
   b) $\mathrm{E}[S_4|N(1) = 2]$
   c) $\mathrm{E}[N(4) - N(2)|N(1) = 3]$

3. There is a sliding door in front of a shop. When customer arrives in front of a closed door it takes S seconds before the door opens. During that time a queue builds up in front of the door. When the door opens all the customers in the queue enter the door at the same time. A timer closing the door gets reseted everytime a customer goes in, and the door is closed if during T seconds no one goes in. Both S and T are some given constants. Furthermore, the interarrival time of customer is assumed to obey Exp(λ)-distribution.

   a) How many customers on average goes through the door from the time it opens to the time it closes?
   b) What is the probability that an arriving customer has to wait in the queue?

4. Service requests arrive at a server according to a Poisson process with intensity λ. If the server is overloaded, its throughput collapses. To prevent this, congestion control based on *gapping* is applied: after every admitted request, new requests are blocked for time T. Assume that such blocked requests do not result in retrials. What is the rate of admitting requests? In particular, what is this rate in the limits where T is either very small or very large?

5. It has been observed that in the interval $(0, t)$ one arrival has occurred from a Poisson process, i.e. $N(0, t) = 1$. Prove that conditioned on this information, the arrival time τ is uniformly distributed in $(0, t)$. Hint: determine the conditional cumulative distribution function of the arrival time $\mathrm{P}\{\tau \leq s \mid \text{one arrival during } (0, t)\}$.

6. Customers arrive at a queue according to a Poisson process with intensity λ. Let X denote the service time of a random customer $X^*(s)$ the Laplace transform of its probability density function. Consider the number of new customers N that arrive at the queue during the service time X.

   a) Derive the generating function $N(z)$ of N by conditioning on the values of X, $N(z) = \mathrm{E}\left[z^N\right] = \mathrm{E}\left[\mathrm{E}\left[z^N|X\right]\right]$.

   b) Apply the previous result to the case $X \sim$ Exp(μ). Show that N has a geometric distribution.

   c) Derive the same result by reasoning. Hint: What is the probability that the next event is i) the arrival of a new customer ii) service completion?