HELSINKI UNIVERSITY OF TECHNOLOGY
Networking Laboratory
S-38.3143 Queueing Theory, II/2007

Exercise 3
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1. Determine the probability distribution in equilibrium for birth-death processes (state space $i = 0, 1, 2, \ldots$), which transition intensities are a) $\lambda_i = \lambda$, $\mu_i = i\mu$, b) $\lambda_i = \lambda/(i+1)$, $\mu_i = \mu$, where λ and μ are constants.

2. In a game audio signals arrive in the interval $(0, T)$ according to a Poisson process with rate λ, where $T > 1/\lambda$. The player wins only if there will be at least one audio signal in that interval and he pushes a button (only one push allowed) upon the last of the signals. The player uses the following strategy: he pushes the button upon the arrival of the first (if any) signal after a fixed time $s \leq T$.

   a) What is the probability that the player wins?

   b) What value of s maximizes the probability of winning, and what is the probability in this case?

3. It has been observed that in the interval $(0, t)$ one arrival has occurred from a Poisson process, i.e. $N(0, t) = 1$. Prove that conditioned on this information, the arrival time τ is uniformly distributed in $(0, t)$. Hint: determine the conditional cumulative distribution function of the arrival time $\mathrm{P}\{\tau \leq s \mid \text{one arrival during } (0, t)\}$.

4. Customers arrive at the system according to a Poisson process with the intensity λ. Each customer brings in a revenue Y (independently of other customers), which is assumed be an integer with the distribution $p_i = \mathrm{P}\{Y = i\},\ i = 1, 2, \ldots$. Let X_t denote the total revenue gained during the time interval $(0, t)$.

   a) Derive expressions for $\mathrm{E}[X_t]$ and $\mathrm{V}[X_t]$.

   b) Deduce that $X_t \sim E_1 + 2E_2 + 3E_3 + \cdots$, where the E_i are independent random variables with the distributions $E_i \sim \mathrm{Poisson}(p_i \lambda t)$.

5. Million (10^6) data packets per second arrive at a network from different sources. The lengths of routes, defined by the source and destination addresses, vary considerably. The time a packet spends in the network depends on the length of the route, but also on the congestion of the network. The distribution of the time a packet spends in the network is assumed to have the following distribution: 1 ms (90 %), 10 ms (7 %), 100 ms (3 %). How many packets there are in the network on average?

6. The states of some irreducible Markov-process, which steady state probabilities π_i are assumed to be known, can be partioned into two disjoint sets, $A = \{1, 2, \ldots, n\}$ and $B = \{n+1, n+2, \ldots\}$, so that the transition rates between the sets are non-zero only from state n to state $n+1$ and the opposite direction, i.e. $q_{n,n+1} = \lambda$ and $q_{n+1,n} = \mu$.

   Using the Little's result write down an equation for the average time it takes from the transition $n \to n+1$ to the time the system returns back to set A (transition $n+1 \to n$), i.e. for the average time the system spends at set B.