Helsinki University of Technology
Signal Processing Laboratory

S-38.411 Signal Processing in Telecommunications I

Spring 2000

Lecture 5: Optimal linear equalizers for linear channels 2

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Timetable

L1 Introduction; models for channels and communication systems
L2 Channel capacity
L3 Transmit and receive filters for bandlimited AWGN channels
L4 Optimal linear equalizers for linear channels 1
L5 Optimal linear equalizers for linear channels 2
L6 Adaptive equalizers 1
L7 Adaptive equalizers 2
L8 Nonlinear receivers 1: DFE equalizers
L9 Nonlinear receivers 2: Viterbi algorithm
L10 GL1: DSP for Fixed Networks / *Matti Lehtimäki, Nokia Networks*
L11 GL2: DSP for Digital Subscriber Lines / *Janne Väänänen, Tellabs*
L12 GL3: DSP for CDMA Mobile Systems / *Kari Kalliojärvi, NRC*
L13 Course review, questions, feedback
E 24.5. (Wed) 9-12 S4 **Exam**

Contents of Lecture 3

Optimal linear equalizers for linear channels 2

I. MMSE equalization

II. Discrete-time FIR equalizers

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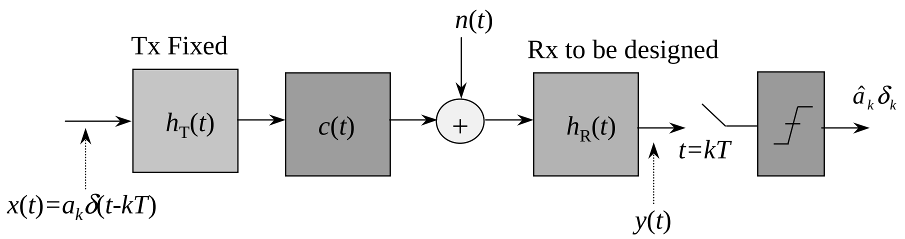
I. MMSE equalization

MMSE equalization

- In the previous lecture, we considered optimization of transmit and receive filters in linear channels with noise
- Remaining problems:
  - usually only Rx filter can be optimized in practice
  - simultaneous noise minimization (Matched filter) and ISI elimination (Nyquist criterion) is not possible at the receiver
  - Zero-forcing (ZF) equalizer removes ISI, but has noise problems
- A new design criterion needed which allows for a *compromise* between ISI and noise

MMSE equalization...

- System model:



MMSE equalization...

- Notation: $x(t) = \sum_k a_k \delta(t - kT), \quad r(t) = h_T(t) * c(t) * x(t) + n(t)$

$$y(t) = h_R(t) * r(t) = h_R(t) * c(t) * h_T(t) * x(t) + h_R(t) * n(t)$$
$$\equiv g(t) * x(t) + n_R(t)$$

$x(t)$ = input signal (symbol sequence)

$h_T(t)$, $h_R(t)$ = transmit and receive filters

$c(t)$ = channel impulse response

$n(t)$ = additive Gaussian noise (colored): $S_n(f) = \frac{N_0}{2} S_{n0}(f)$

Normalization: $\int_{-\infty}^{\infty} S_{n0}(f)\,df = 2$

MMSE equalization...

Assumptions:

- Tx filter $h_T(t)$ is fixed
- Rx filter $h_R(t)$ is to be optimized
- Optimization criterion: *minimum total MSE*
- Channel $c(t)$ and noise power spectrum $S_n(f)$ known at Rx

MMSE equalization...

- Ideally, with no ISI and no noise, the signal after Rx is as desired (at sampling instants). Otherwise there is an error:

$$\begin{aligned} e(t) &= y(t) - x(t) \\ &= [h_R(t) * c(t) * h_T(t) - \delta(t)] * x(t) + h_R(t) * n(t) \end{aligned}$$

- Minimize MSE:

$$\mathrm{E}[e^2(t)] = \mathrm{E}[\{h_R(t) * c(t) * h_T(t) - \delta(t)\} * x(t) + h_R(t) * n(t)]^2$$

MMSE equalization...

- MSE via error power spectrum:

$$\mathrm{E}[e^2(t)] = \int_{-\infty}^{\infty} S_e(f)\,df, \qquad S_e(f) = \int_{-\infty}^{\infty} r_e(\tau) e^{-j2\pi f\tau} d\tau$$

- Assume that signal and noise are *independent*

$$\Rightarrow \; S_e(f) = |H_T(f)C(f)H_R(f) - 1|^2 S_x(f) + |H_R(f)|^2 S_n(f)$$

MMSE equalization...

- Combine Rx terms (to be solved!) and complete the square:

$$S_e(f) = S_r(f)\left|H_R(f) - H_{TC}^*(f)S_x(f)/S_r(f)\right|^2 + S_x(f)S_n(f)/S_r(f)$$

where

$$H_{TC}(f) = H_T(f)C(f)$$

$$S_r(f) = \left|H_{TC}(f)\right|^2 S_x(f) + S_n(f)$$

- How to minimize the MSE?

MMSE equalization...

- Set first term to zero:
  (→ MSE = MIN at any frequency → total MSE = MIN)

$$S_r(f)\left|H_R(f) - H_{TC}^*(f)S_x(f)/S_r(f)\right|^2 = 0$$

- Optimal MSE solution for Rx filter (= MMSE equalizer):

$$H_R(f) = \frac{H_{TC}^*(f)S_x(f)}{S_r(f)}$$

$$= \frac{H_T^*(f)C^*(f)S_x(f)}{\left|H_T(f)C(f)\right|^2 S_x(f) + S_n(f)}$$

MMSE equalization...

Observations on the MMSE solution:

1) High-noise case:

$$H_{\mathrm{R}}(f) \approx \frac{H_{\mathrm{T}}^{*}(f)C^{*}(f)S_{x}(f)}{S_{n}(f)}$$

- Matched filter!

2) High-ISI case:

$$H_{\mathrm{R}}(f) \approx \frac{1}{H_{\mathrm{T}}(f)C(f)}$$

- Zero-forcing equalizer!

MMSE equalization...

The MMSE solution:

- enables a good compromise between noise and ISI minimization
- is robust (avoids problems with channel zeroes)
- is widely used in practice
- enables efficient adaptive implementations
  (with discrete time FIR filters!)

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II. Discrete-time FIR equalizers

Discrete-time FIR equalizers

The previous equalizers' impulse response is

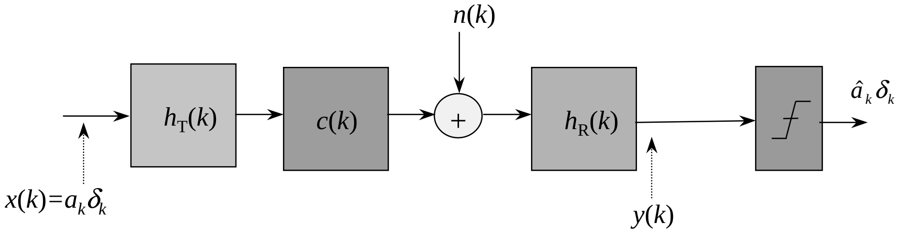
- continuous-time
- infinite-length
- non-causal

Practical equalizer implemented almost always with discrete-time Finite Impulse Response (FIR) filters

- finite complexity
- causal
- can be made adaptive with simple methods

Discrete-time FIR equalizers...

- Discrete-time system model:
  Sampling at symbol rate ($f_S = 1/T$, $x(k)=x(kT)$ etc.)



Discrete-time FIR equalizers...

Discrete-time notation:

$$x(k) = \sum_k a_k \delta_k$$

$$r(k) = h_T(k) * c(k) * x(k) + n(k)$$

$$y(k) = h_R(k) * r(k)$$

$$= h_R(k) * c(k) * h_T(k) * x(k) + h_R(k) * n(k)$$

$$= g(k) * x(k) + n_R(k)$$

Note! Because of symbol-rate sampling some aliasing may happen (excess bandwidth!)

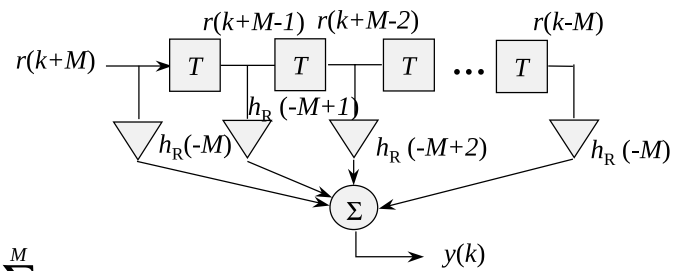
Discrete-time FIR equalizers...

- The previous continuous-time equalizers can be rederived for discrete-time infinite-length filters
- Instead, let us go directly for discrete-time FIR filters!
- Receive filtering:

$$
\begin{aligned}
y(k) &= h_{\mathrm{R}}(k) * r(k) \\
&= \sum_{l=-M}^{M} h_{\mathrm{R}}(l) r(k-l)
\end{aligned}
$$

Discrete-time FIR equalizers...

- FIR filter structure:



$$y(k) = \sum_{l=-M}^{M} h_{\mathrm{R}}(l) r(k-l)$$

Discrete-time FIR equalizers...

- Convolution in vector notation:

$$y(k) = h_{\mathrm{R}}(k) * r(k) = \sum_{l=-M}^{M} h_{\mathrm{R}}(l) r(k-l) = \mathbf{h}_{\mathrm{R}}^{\mathrm{T}} \mathbf{r}(k)$$

- FIR coefficient and signal vectors:

$$\mathbf{h}_{\mathrm{R}} = \left[h_{\mathrm{R}}(-M) \quad \cdots \quad h_{\mathrm{R}}(M-1) \quad h_{\mathrm{R}}(M) \right]^{\mathrm{T}}$$

$$\mathbf{r}(k) = \left[r(k+M) \quad \cdots \quad r(k-M+1) \quad r(k-M) \right]^{\mathrm{T}}$$

Discrete-time FIR equalizers...

- Consider *k*th error signal sample:

$$e(k) = y(k) - x(k) = \mathbf{h}_{\mathrm{R}}^{\mathrm{T}} \mathbf{r}(k) - a_k$$

- The MSE in vector notation:

$$\mathrm{E}\left[e^2(k) \right] = \mathrm{E}\left[\left(\mathbf{h}_{\mathrm{R}}^{\mathrm{T}} \mathbf{r}(k) - a_k \right)^2 \right]$$

Discrete-time FIR equalizers...

- Elaborate the MSE:

$$\begin{aligned}\mathrm{E}\left[e^2(k)\right] &= \mathrm{E}\left[\left(\mathbf{h}_\mathrm{R}^\mathrm{T}\mathbf{r}(k) - a_k\right)^2\right] \\ &= \mathrm{E}\left[a_k^2\right] - 2\mathbf{h}_\mathrm{R}^\mathrm{T}\mathrm{E}\left[\mathbf{r}(k)a_k\right] + \mathbf{h}_\mathrm{R}^\mathrm{T}\mathrm{E}\left[\mathbf{r}(k)\mathbf{r}^\mathrm{T}(k)\right]\mathbf{h}_\mathrm{R} \\ &= \mathrm{E}\left[a_k^2\right] - 2\mathbf{h}_\mathrm{R}^\mathrm{T}\mathbf{p} + \mathbf{h}_\mathrm{R}^\mathrm{T}\mathbf{R}\mathbf{h}_\mathrm{R}\end{aligned}$$

- Define *autocorrelation matrix*:

$$\mathbf{R} = \mathrm{E}\left[\mathbf{r}(k)\mathbf{r}^\mathrm{T}(k)\right]$$

and *crosscorrelation vector*:

$$\mathbf{p} = \mathrm{E}\left[\mathbf{r}(k)a_k\right]$$

Discrete-time FIR equalizers...

- Example: consider 1-tap FIR filter (1 coefficient only):

$$\mathbf{h}_\mathrm{R} = h_\mathrm{R}(0) \qquad \mathbf{R} = \mathrm{E}\left[r^2(k)\right] = R_0 \qquad \mathbf{p} = \mathrm{E}\left[r(k)a_k\right] = p_0$$

- Elaborate MSE (complete the square!):

$$\begin{aligned}\mathrm{E}\left[e^2(k)\right] &= \mathrm{E}\left[a_k^2\right] - 2h_\mathrm{R}(0)p_0 + R_0 h_\mathrm{R}^2(0) \\ &= \mathrm{E}\left[a_k^2\right] - \frac{p_0^2}{R_0^2} + R_0\left(h_\mathrm{R}(0) - \frac{p_0}{R_0}\right)^2\end{aligned}$$

Discrete-time FIR equalizers...

- Optimal MMSE solution: set square term to zero:

$$h_{\mathrm{R}}(0) = \frac{p_0}{R_0}$$

- Minimum MSE:

$$\mathrm{E}\left[e^2(k)\right] = \mathrm{E}\left[a_k^2\right] - \frac{p_0^2}{R_0^2}$$

Discrete-time FIR equalizers...

- General case of N-tap FIR filter:

$$\begin{aligned}\mathrm{E}\left[e^2(k)\right] &= \mathrm{E}\left[a_k^2\right] - 2\mathbf{h}_{\mathrm{R}}^{\mathrm{T}}\mathbf{p} + \mathbf{h}_{\mathrm{R}}^{\mathrm{T}}\mathbf{R}\mathbf{h}_{\mathrm{R}} \\ &= \mathrm{E}\left[a_k^2\right] - \mathbf{p}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{p} + \left(\mathbf{h}_{\mathrm{R}} - \mathbf{R}^{-1}\mathbf{p}\right)^{\mathrm{T}}\mathbf{R}\left(\mathbf{h}_{\mathrm{R}} - \mathbf{R}^{-1}\mathbf{p}\right).\end{aligned}$$

- MMSE solution:

$$\mathbf{h}_{\mathrm{R}} = \mathbf{R}^{-1}\mathbf{p}$$

Discrete-time FIR equalizers...

Properties of the MMSE solution:

- autocorrelation matrix **R** always positive (semi)definite

  => unique minimum always exists (error surface is *N*-dimensional paraboloid, 'bowl')
- on-line solution requires estimation of **R** and **p** & matrix inversion => computationally intensive, numerical problems
- iterative solutions for the matrix inversion

  => adaptive filter theory!

Summary

Today we discussed:

Optimal linear equalizers for linear channels 2

I. MMSE equalization

II. Discrete-time FIR equalizers

Next week: Adaptive equalizers I